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# Socializing Mathematical Instruction[1]

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THE statement that all men are created equal has all too often been interpreted to mean that there is only a sameness to humanity, and hence that all men are to have the same of everything in life, the same worldly goods, the same schooling, the same recreation, the same rights and privileges, the same mathematical education. Of course we know this is nonsense, recognizing that all of us differ in endowed talents, in degrees of performance, and in the types of instruction and schooling we should obtain. We must always remember that the essence of democracy is difference, not sameness, and that our schools must provide for this difference. Yet to preserve our democracy we must share a common heritage, a sameness that unites us as one nation, and we must likewise provide for this sameness in our schools. It is in this light that a mathematics program must be devised for the oncoming generation.

At present, in general, the mathematical program of the first eight school years contains that knowledge which is essential to the every-day living of every citizen. It is a common core of knowledge adapted to the sameness of our society. Arithmetic skills and arithmetic reasoning are developed, improved, and extended. Arithmetic is applied to problems of the home, of business, of insurance, of savings, of investments, and of the shop. Practical geometry is taught as applied in measurement, design, scale drawing and elementary surveying. The formula, the simple equation, and the graph are taught as they are found used in the public press. The entire emphasis is on application and use in life, not on mathematics for its own sake.

In the ninth and tenth years, to a large extent, we continue the education for sameness, and at the same time provide the beginning of education for individual differences. In ninth grade algebra, the core is the fundamental method of expressing relationships between quantities, including the rule, the formula, the equation, the graph, and the table of values. At the same time we develop the laws of general number, we extend the number system to include new types of numbers, we generalize laws for treating products, factors, fractions, and other concepts necessary for the development of abstract algebraic study. The way pupils react to these two distinct presentations gives evidence of their interests and their capacities, and this evidence should be used in guiding the pupils into senior high school study adapted to their differences.

In the tenth year, the work in plane ge-

[1] An address, delivered at Teachers College Columbia University, Thursday July 24, 1947.

ometry continues the common core knowledge by presenting "methods of thinking." Methods of proof; types of reasoning; and applications of the reasoning and derived facts to everyday social behavior are or should be a large part of the course. At the same time there is developed the nature of a mathematical system; proofs of theorems and of original problems are demanded that are necessary for an insight into, and a continued study of, advanced mathematics. The reaction of pupils to their study of geometry gives added information regarding their abilities and interests in mathematical study. In the ninth and tenth years, a small number of schools offer a general mathematics designed to meet the individual differences of the less mathematically inclined pupils.

The eleventh and twelfth years provide a mathematical education designed only for one purpose, to meet college entrance requirements. For the most part these courses are not even designed to cover the individual differences of the pupils taking them, but are geared to an average college preparatory level, that becomes lower as more pupils seek to enter college. This lower level is frightening many college mathematics professors today who are insisting that we teach only the fundamentals of algebra and geometry, and do it more thoroughly and with more drill. Their only desire is that we teach to develop pure mathematicians; and those pupils who do not profit by such instruction are of no concern to them. Because of this, the general educator says rightly, that the mathematics instruction of the senior high school is entirely mathematics for its own sake, unrelated to the pupil's life or needs, and that such instruction has very little value. What stand are we, the teachers of mathematics, to take?

A large group of educators is calling for a complete socialization of the high school program by the use of a core curriculum. This curriculum would take the common experiences of all fields of subject matter and combine them into an "integrated program" in which the pupil will learn by using all knowledge indiscriminately. Most forward looking mathematics teachers agree that we must stop compartmentalizing our mathematics by subjects and years, and proceed to fuse algebra, geometry, trigonometry and analysis into a single complete development of the science of number and of space. But the core curriculum goes one better and includes not only this, but a fusing of all subjects into one single complete development of knowledge that will function in the life of every individual.

Mathematics teachers are afraid of this, because they are afraid the good ship mathematics may be lost in the process. The educators who are proposing the core curriculum are intelligent men and in seeking an answer to their proposal we can well follow the advice of Bertrand Russell,[2] who said "Two things are to be remembered: that a man whose opinions and theories are worth studying may be presumed to have had some intelligence, but that no man is likely to have arrived at complete and final truth on any subject whatsoever. When an intelligent man expresses a view which seems to us obviously absurd, we should not attempt to prove that it is somehow true, but we should try to understand how it ever came to *seem* true. This exercise of historical and psychological imagination at once enlarges the scope of our thinking and helps us to realize how foolish many of our own cherished prejudices will seem to an age which has a different temper of mind."

Let us then be perfectly frank and seek to learn if socialization is a desirable program in the high school, and if it is, how it can be achieved. First, is there any tie-up of the mathematical instruction in any given year, say the ninth year, with the material presented in the other subjects? Has any attempt been made to present that mathematics in the ninth year that can be used in solving problems in the

[2]Bertrand Russell, ***A History of Western Philosophy.*** Simon and Schuster, 1945. p. 39.

civics class, problems relating to government financing, town surveying, housing, etc? Do you know what is taught in ninth year general science, and do you attempt to teach the algebra that could be used to help the pupil in his study of science? Have the other departments made any attempt to cooperate with the mathematics department in working out a coordinated program? I can point to isolated experiments carried out in the 8th and 12th grades over a number of years, where a coordinated program of the science, social studies, English and mathematics department, resulted not only in a more vitalized program, but in no loss of pure mathematics progress whatsoever. Does this not seem to indicate something in favor of the core curriculum?

In this connection I should like to mention Dr. Condon's study on the mathematics needed in the study of physics. Dr. Condon did both a favor and a disfavor to mathematical education by his study. First he showed that for a general knowledge of physical concepts, there is needed only a minimum of mathematical knowledge that can be taught in the high school 9th and 10th years. In this day when physics is such an important part of our everyday experience Dr. Condon showed that a very elementary knowledge of mathematics is sufficient to endow all citizens with a necessary common core of knowledge. On the other hand, many educators, not versed in physics and mathematics, have interpreted Dr. Condon's conclusions to mean only a slight knowledge of mathematics is necessary to study *any* physics. Nothing is further from the truth. The study of scientific physics today demands not only complicated algebra and geometry, but a thorough knowledge of analytics and the calculus. To study "Freshman Physics" in our Technology Institutes, the student must understand the elements of calculus.

Secondly, do we have any provision in our courses for experimentation and the development of the scientific method? Do we combine with the art department in the study of dynamic symmetry and its use in beautifying our surroundings? Do we combine with the social studies department in a project of collecting data and analyzing the data to determine how much it costs to rear a child to the age of 18 years? Or, which is better, a Ford car worth \$1500 at graduation from high school or a \$500 annuity to begin at age 50 years? Do we provide a laboratory where the application of geometric principles to machines can be studied at first hand? And would such studies be worth more to the greater majority of our pupils, than the present exposition of algebra and geometry?

Thirdly, do we use the library, books other than our texts, the newspaper, and pamphlets to enrich and broaden the mathematical knowledge and to show its relation to other fields of knowledge? The English teachers do, the social study teachers do, and the science teachers do. When a group of mathematics students show up in a library they usually argue over the solution of an assigned problem in their text, eventually becoming so noisy that the librarian orders them out.

Finally, many teachers of mathematics take further study in courses showing the uses and applications of mathematics in the other fields of knowledge, thereby enriching their own knowledge and understanding. Do they apply this to their teaching? In most cases, they return to their schools and teach the same old mathematics in the same old way, because the head of the department will not permit them to use these "new-fangled ideas," or because "there is no money to buy simple mathematical instruments," or because they cannot change their classroom or take a class out of doors.

We must admit that the educators have a strong point to their criticism of the present instruction in mathematics. We can not admit, however, that they have the best answer for improvement in their unified curriculum of all knowledge. Let us see why not.

The fundamental aim of mathematics instruction was exceptionally well stated in the *Reorganization of Mathematics in Secondary Education* report of 1923. Here is the statement:

The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space, which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects; and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

"That is exactly what we want," says the educator. Will our students have an insight into and control over their environment under our present instruction? Does their present mathematical instruction give them an appreciation of the progress of civilization in its various aspects? Frankly, I doubt it.

Other worthy objectives of mathematical instruction are given in the Fifteenth Yearbook of the National Council of Teachers of Mathematics. They are only four in number: to develop the ability to think clearly; to develop fundamental skills, knowledge, and concepts; to develop desirable attitudes; and to initiate and develop interests and appreciations in the humanities, arts and sciences that make for cultural growth, and richness of living. Do you believe that our present mathematical instruction attains to any marked degree, any of these objectives, except perhaps that of fundamental skills? Frankly, I doubt it.

It cannot be denied that mathematics is a large integral part of our modern culture. The humanitarians and educators know this, and what worries many of them is that they do not understand the real nature of mathematics because they have not studied it sufficiently. They are therefore unable to see its place in the total program of education, although they sense the need. Today there is hardly a field of knowledge that has not been impregnated with mathematical modes of exposition. Yet the high school teachers in these other fields of knowledge are for the most part ignorant of mathematics and mathematical modes, and hence avoid all mention of mathematics in the subjects they teach. Many of these teachers are afraid of mathematics.

*Until the teachers of English, social studies, science, art, music, etc., are required to study sufficient mathematics to understand its basic principles, and its important role in their own field and in modern culture, the socialization of mathematical knowledge must be carried out by the mathematics teachers in their own classes.*

The past training of teachers further justifies this procedure. In teachers colleges, there is no required mathematics instruction, or at most a single course on arithmetic. On the other hand, all undergraduates are required to study English, social studies, physical and biological sciences for 6, 12 or 18 semester-hours. Hence mathematics teachers have at least a background in the other subjects that the other teachers lack in mathematics. If we, the teachers of mathematics, carry out a program of socialization of our subject, we render a service not only to the furtherance of functional education but also to the enriched instruction of mathematical skills and meanings necessary for the preservation of our democratic socicty.

What shall we do, to initiate a program of socialization? First get acquainted with the Seventeenth Yearbook of the National Council of Teachers of Mathematics which shows how algebra, geometry and trigonometry are applied in various other fields. This is a start. But what is needed is a new yearbook which shows the mathematics that can be applied in the other fields, on the high school level. We need a study on the mathematics in economics, the mathematics in physics, the mathematics in linguistics, in geography, in art and music, in biology, in chemistry, in astronomy, in business, and so on. Here is a real opportunity for research and progress.

In order that a teacher may recognize

and be able to use these applications in other fields, there are two pre-requisites. (1) Every teacher must have a complete and thorough knowledge of mathematics at least through the calculus and beyond if possible. This must include a real understanding of the foundations and logic of mathematics. It should without doubt include a thorough history of the development and applications of mathematics. To teach mathematics as a logical system along with its social uses, necessitates better teaching which in turn necessitates a real knowledge of what is and what is not essential in mathematics study.

Then we must make it our business to know what is being taught in the high school, in the other subject matter fields, in each year of study. Too frequently we are concerned only with the mathematics the pupils had before they came to our class, and the mathematics they will take the next year, ignoring all the other subject matter fields. Let us cease teaching mathematics for its own sake only, and for College Entrance. Let us not render lip service alone, but actually take the initiative in the act of socialization. The other subject teachers may be frightened at first, but they will have to fall in line with a socialization program. And the easy thing about all this is that in any year, no matter what mathematics you are teaching, you can always find an application of it in the other fields that are being taught then.

Today, many schools have introduced, and many more schools will be introducing, a core program of education in the secondary school instruction. If mathematics is to maintain its proper and necessary function in general education, the mathematics teachers in these schools must cooperate with the program. Not only must they cooperate, but they must broaden their own knowledge so that they can contribute wisely and authoritatively to the proper execution of a core curriculum. They must have all the background previously referred to in this article, along with the exercise of keen judgement as to what is and what is not a reasonable adaptation of mathematics to the various phases of the program. It should always be recalled that the core curriculum does not eliminate separate sequential mathematical study which can be elected by those capable, interested or in need of such study. The core curriculum brings in only that mathematics which is needed by all members of a democracy for everyday intelligent evaluation of their lives. The specialized mathematics for particular vocational or career needs is taught as an elective, simultaneously with it.

Finally, it must be remembered that mathematics is a discipline that can be had only through systematic treatment with proper drill and problem solving. We cannot turn our mathematics classroom into a continuous laboratory for discovering facts or making interesting applications. But we can spare at least one lesson a week in meeting the primary purpose of the teaching of mathematics. Let us start by asking the pupils to make reports on the mathematics they are studying and its relation to the other subject. Let us collect a library of mathematics books in our school for pupil use, that will enable them to extend, apply, and appreciate what they are learning in class. Let us build a laboratory in our classroom of models, visual aids, and devices showing the applications of mathematics. We can rest assured that the results of one lesson a week, (not necessarily every week, or the same day each week,) spent on making mathematics alive, useful and meaningful, will soon speed up the progress that is made in the straight mathematics instruction, so that at the end of the year we will be farther along in our instruction than if we had omitted the socialization.

We have an opportunity to enrich our instruction for better living and understanding of life, or we can continue to teach so that adults will say: "I never used any of the algebra or geometry taught to me in secondary school, and I see no reason why my children should study it."

# A Core Curriculum for the Training of Teachers of Secondary Mathematics

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THE following discussion is planned to outline the complete education of teachers of secondary mathematics in a four-year program, leading to a baccaleureate in a school of education, college of liberal arts, or teachers college. It is based upon many of the past reports and articles concerning the training of mathematics teachers, and gives a summary of past recommendations as well as a total, new four-year plan.

## INTRODUCTION

*THESIS 1: The training of secondary mathematics teachers should include general, mathematical and professional education. No one of these three types of education can be omitted.*

The Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics in their report "The Place of Mathematics in Secondary Education" (Bib. #19) says "there are five major qualities that are to be considered in the mathematics teacher: (1) social and civic attributes, (2) general culture, (3) familiarity with educational problems and theories, (4) skill in instruction, and (5) knowledge of and interest in mathematics." In the discussion that follows, #3 and #4 will be grouped under the general heading of *professional education.*" We shall, of course, include their #5 under mathematical education, and #2 under a general education. There may be some discussion concerning the classification of #1 on their list. We shall take the attitude that social and civic attitudes are best taught by discussing examples, which appear naturally, in the work in general and professional education, rather than as a separate entity or separate course.

At the Institute for Administrative Officers of Higher Institutions, held at the University of Chicago in 1935, Hayward Keniston spoke on "General and Liberal Education in the Preparation of Teachers" (Bib. #22, page 164). He asked the basic question of the present discussion: "These two elements of special training and of general education are of equal importance in the training of the teacher; either one will remain ineffective and hollow without the other. How shall a prospective teacher maintain a proper balance between the two?"

*THESIS 2: The relative amounts of time devoted to general, mathematical and professional education should be determined by the needs of the teachers-in-training and not by traditional courses.*

Most of the present curricula for college courses, teacher-training or otherwise, are created by piecing together half-courses, or semester courses, which have been kept by tradition. There is so much demand for the time of the students that these courses should be examined to see whether they fit the objectives set. The relative importance of the three major types of education in teacher-training, and the importance of the various sub-objectives should first be determined and then time allotted accordingly. We are demanding that the courses and curriculum on a secondary level remain flexible and constantly adjust themselves to the changing needs of the pupils; why should not the collegiate courses as well.

*THESIS 3: On the basis of a four-year program of 120 semester-hours, a desirable distribution of time would be as follows: General Education—50 hours (30 hours prescribed, 20 hours free elective), Mathematical Education—40 hours, and Professional Education —30 hours.*

The Cooperative Committee on Science Teaching of the American Association for the Advancement of Science is studying the problem of preparation of science teachers, including mathematics, and have already published some suggestions. In 1946 (Bib. #11) they made the following three points: (1) Teachers should be prepared in three of the five following fields—biological sciences, chemistry, mathematics, physics and general science, (2) 60 semester hours should be devoted to science (24 to a major study and 18 to each of two minors) and (3) Teaching training should work toward a five-year program. This program may serve to create good teachers of science as technicians, but it will not give the schools teachers who are interesting, alive, broad citizens of the world. (See Thesis #4, below). Demanding 60 semester hours within the field of science, even including mathematics, is asking for too much time in proportion to the other objectives of a four-year, teacher-training education. (See Thesis #2, above) Adding the hope that five years will some day be available for this training does not excuse the usurpation of the time in a four-year program. Much as it may seem like beating on my own little drum, I do not believe that mathematics belongs in the group of five sciences listed. It still has its place and importance in the world, and therefore in school, comparable to *Science* as a whole, and not the various subdivisions of science.

Bagley in 1933 (Bib. #1) tried to settle the division of hours by first dividing the work into professional content courses, professional background courses, and education courses. It is easy to detect the interest he had in professionalized subject-matter courses in his excessive use of the word "professional" in this division. There would be 60 semester-hours of professional content courses; 30 in mathematics, 30 in other fields of specialization. In professional background courses, 40 semester hours. These are sometimes called cultural courses or integrated courses; in this article they are called general education. He also says education courses should include at most 10 or 12 semester-hours, exclusive of contact with laboratory schools. The allotment to general education is a little low; if some general education courses are rigidly prescribed (See Theses #5 & 6, below) it will be necessary to allow more room for free choice to meet individual needs. Part of this allotment can be used to bolster teaching minors as needed. The allotment for professional education will be found to be much too small once we start to design definite courses. (See Thesis #11, below).

### General Education

*THESIS 4: Mathematics teachers must be trained* primarily *as citizens of a democracy to have well-rounded, interesting personalities, and* secondarily *as members of their profession.*

The Commission on Training and Utilization (Bib. #9, pg. 272) begins its discussion of undergraduate training for teaching secondary mathematics with the following paragraph:

> A Teacher, to be of maximum service to the community in which he lives, should be recognized as an educated man to whom adult members of the community may turn for consultation on intellectual affairs. He should be able to participate in community activities, and assume his share of leadership. Certainly he cannot function satisfactorily if he is notably ignorant of what are commonly regarded as fundamentals of general culture. With these facts in mind we advocate a breadth of training for teachers of mathematics which will insure a degree of familiarity with language, literature, fine arts, natural science, and social science, as well as mathematics.

In teacher personality, outlook on life, culture, and character are just as important as knowledge of facts and ideas. The selection and method of presentation of material which teachers use are colored by their personal characteristics. Also, the examples which they set by their own conduct is a very effective part of their teaching. Many of them complain that they do not want the community to interfere with their private lives. This reminds one of

movie stars who say they do not like publicity; they should have thought of it earlier. Both the children and their parents expect teachers to be worthwhile citizens and interesting people as well as dealers in subject-matter.

In greeting the members of the Institute for Administrators of Higher Institutions (Bib. #13, Page 3), President Hutchins hit the nail right on the head: "Most of the argument about teacher training seems to be beside the point. The argument revolves about the question whether a prospective teacher should take a lot of courses in education or a lot of courses in subject matter. The answer is that the teacher should understand education and should understand his subject, but first of all he should himself be educated. He should have a good general education."

A recent article by Philip Wylie in *This Week* called "What's Your Diploma Worth?" (Bib. #39) contains two paragraphs "boosting" general education:

> Our schools increasingly regard the teaching of a trade as their chief function. Any fool can learn a trade, but knowing a trade has nothing to do with being an educated, competent citizen in a republic! So, as schools grow more and more 'vocational,' their moral authority and intellectual prestige diminish.
>
> . . . . . . . . . . . . . . . . . . . .
>
> The educated man is mature. His maturity is based on articulation; and understanding of general principles, not gadgets; and on truth, not boosterism. The educated man seeks more truth—more education all his life. His life of humanity is not sentimental but honest, and he expresses it by the way he acts. He can be trusted. And he serves himself by serving liberty, which is his fundamental principle.

If we overlook the half-truths and vague words in parts of these paragraphs we will find that there still remains a great deal of truth in what he says.

This problem is not new. In 1933, the National Survey of the Education of Teachers (Bib. #28, Vol. 3, p. 200) said that the problem of the basic education of American teachers "is in part if not largely identical with the problem of the liberal education for all college students."

*THESIS 5: One part of the required, general education should provide the background of information, ideas and attitudes needed by all educated people to talk, read and write with their fellow men. The following areas should be covered: The Humanities, The Social Studies, and The Sciences.*

The National Survey of the Education of Teachers (Bib. #28) determined the extent to which prospective teachers had had contact with certain areas important in the background of educated people. The results follow: Graphic Arts—37%; Music—30%; Economics—44%; Political Science—57%; Sociology—54%; and Biological Sciences—82%. Should not all who receive a college education have had some contact with each of these subjects?

What criterion should determine the subject-matter chosen for general education? Keniston (Bib. #22, page 165) seems to think that it makes little difference what the subject-matter is, for he says: "—it is not the content of the subject which is the determining factor, but rather the degree to which the subject offers possibilities of widening the mental outlook and experiences of the student." This attitude assumes that it would be impossible to widen the outlook and experience and at the same time provide subject-matter material which will be of lasting use and importance. Why not ask for topics which contribute both information and attitudes?

How have liberal arts colleges solved the problem of choosing material for general education which meets the standard set above? Harvard and Columbia Colleges have received most publicity for their programs. Harvard (Bib. #32) requires six courses in general education, at least one in each of the following fields: The Humanities, The Social Sciences, and The Sciences. Under The Humanities is a course in "Great Texts in Literature." Other topics which will be included in other courses include literature philosophy, fine arts, and music, The Social Sciences has a course called "Western Thought and Institutions," and will have others on the subjects of American De-

**Table I**

*Suggested Courses in General Education*

| Course | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Humanities | — | — | — | — | — | — | R | R |
| Philosophy | D | — | — | D | — | D | — | — |
| Religion | — | — | — | D | — | — | — | — |
| Ethics | D | — | — | D | — | — | — | — |
| Logic | — | — | — | — | — | D | — | — |
| Fine Arts | — | D | D | — | — | D | — | — |
| Art | — | — | — | D | — | — | — | — |
| Music | — | — | — | D | — | — | — | — |
| Literature | — | D | D | D | — | — | — | — |
| Languages | — | D | D | — | — | D | — | R |
| Public Speaking | — | — | — | — | D | D | — | — |
| Social Sciences | — | D | D | D | — | R | R | — |
| Political Science | D | — | — | — | — | — | — | — |
| Contemporary Civilization | — | — | — | — | — | — | — | R |
| History | D | — | — | — | D | — | — | — |
| Sociology | D | — | — | — | — | — | — | — |
| Science and Mathematics | — | — | — | — | — | — | R | — |
| Science | — | — | D | — | — | — | — | R |
| Physical Sciences | — | — | — | D | — | — | — | — |
| Biological Sciences | — | — | — | D | — | — | — | — |
| Physiology | — | — | — | — | D | — | — | — |
| English Composition | — | R | — | — | D | R | R | R |

Note: D—Desirable, R—Required.

| Column | Date | Author | Bib. # |
|---|---|---|---|
| 1 | 1923 | Committee on Mathematical Requirements | 26 |
| 2 | 1935 | Commission on Training and Utilization | 9 |
| 3 | 1937 | Knudsen | 23 |
| 4 | 1938 | Hagen and Samuelson | 14 |
| 5 | 1940 | Olds | 30 |
| 6 | 1941 | Boston University, CLA, Dept. of Math. | — |
| 7 | 1945 | Harvard Committee | 32 |
| 8 | 1946 | Columbia Committee | 10 |

mocracy and Human Relations. The Science section assumes that minimal mathematics has been covered at the secondary school level and therefore includes courses in the principles of the physical sciences and one in the principles of the biological sciences.

Columbia College (Bib. #10) also has three divisions of general education: (1) Introduction to Contemporary Civilization, which is offered four times weekly in the freshman year and three in the sophomore year, (2) Introduction to The Humanities, which unites the interests of all the departments concerned with literature, music and the fine arts (First year: Readings in literature, philosophy and history from Homer to Goethe; Second year: A term each of music and fine arts), and (3) Introduction to the Sciences, which is organized around the concepts of matter, energy and radiation; the study of the earth as part of the universe; and the development of plant and animal life.

The requirements of these two colleges together with six other sets of recommendations will be found summarized in table I.

There are two tendencies in the data above which should be noted: (1) The general education is changing from many courses which it would be desirable to have, to a few courses which are *required*, and (2) The type of course which is listed is now a general, survey-type course especially designed for general education

rather than the broadest, most elementary courses that can be found in the usual run of specialized courses.

Consolidating the best features of the plans summarized above we get the following three courses. It is suggested that each be given for a full year, meet three times a week and carry five points, or semester-hours, for the year's work.

***The Humanities***

- **Philosophy: Relationship of man to the Universe; Logic (materials, methods and mechanisms of thinking).**
- **Fine Arts: Literature (Forms of literature, Place of the Classics in Life, Contemporary literature), Graphic Arts (History of art, Analysis of drawing, painting, sculpture and architecture, Use of the graphic arts in education), Music (History of music, Analysis of musical forms, Use of music in education), Motion Pictures (How they are produced, Motion pictures as an art form, Criticism and evaluation of motion pictures, Use of motion pictures in education), Radio Analysis of types of programs, Use of radio in education).**

***The Social Studies***

- **History: Lessons learned from the past, Current affairs.**
- **Government: Responsibilities of the individual citizen, International relations, Comparative government.**
- **Economics: Impact upon personal problems, Relationships between economics and forms of government.**
- **Sociology: Causes of social change, The family as a unit, Group interests and formations, Adjustment of society to environment.**

***The Sciences***

- **Mathematics: Influence on philosophy and other fields of knowledge.**
- **Physics: Matter, energy and radiation; The earth as part of the universe; Nuclear physics and atomic power.**
- **Chemistry: Dyeing and bleaching, foodstuffs, consumer research, fire extinguishers, mending china, fertilizers, insecticides, etc.**
- **Biology: Plant and animal life, The human machine, Evolution, Hygiene and first aid.**

*THESIS 6: One part of the required, general education should provide knowledge of the basic skills which make life easier and more interesting. The following areas should be covered: The Communication Arts, The Home, and Problems in Human Relationships.*

Courses in basic skills of living are seldom offered or required on a college level. Some say that they should be covered on the secondary school level, but this is not facing the facts; they are not covered and they are very much needed. Other courses have originated on the college level and been adapted and adopted by the secondary schools. Even if all secondary schools covered these areas (and some already do) there would still be more advanced, more mature material left for the college level.

The usual courses required in college which fit into this category are the following: English composition, physical education and hygiene, and required foreign languages which have some use as tools. These leave many important skills untouched. What is being done about the speech arts, the problems of marrying and running a home, the problems of human relations outside the home in social contacts and business?

With these questions in mind the following three courses are suggested. Each should be given a full year, meet three times a week and carry five points, or semester-hours, for the year's work.

***Communication Arts***

- **English composition: Written compositions, functional grammar, philology and semantics.**
- **Speech Arts: Public speaking, dramatics, choral speaking, remedial speech.**
- **Foreign Languages: Reading, speaking and writing—One language to be chosen and carried through these skills.**

***Home***

- **Budgeting: Method of planning, Buying.**
- **Home Economics: Cooking, decorating, gardening.**
- **Home Repairs: Building (cement, woodworking, metal work, etc.), Electrical (Toasters, motors), Plumbing (Faucets, sinks, leaks).**
- **Marriage and Family Problems: Choice of a mate, adjustments (temperament, sexual, economic), psychology and training of children.**

***Human Relations***

- **Religious Life: Need for religion, types of religion in the world.**
- **Social Life: Understanding people (Analyzing reasons for reactions, tolerance), Parties (planning, etiquette), Skills (driving car, sailing boat, bridge, swimming, fishing, dancing, horseback riding, etc.)**
- **Business Relationships: Vocational guidance, standards of conduct in the business world.**

*THESIS 7: Some courses in which manual skills are developed should be required.*

College education is almost 100% education of the head and 0% education of the hands. And yet teachers are called upon to draw, manipulate models, arrange their classrooms, mount pictures, operate audio-visual aids projectors, and in general present a picture of a physically coordinated person who does not embarrass the students with awkwardness. The lack of such training is evident every day as experienced teachers approach audio-visual aids with hesitation and even fear.

The six courses outlined above under theses #5 and 6 will provide opportunity for developing manual skills if the college faculties do not think in terms of their own past training and crowd such active learning out of the curriculum. With all the emphasis and recognition of audio-visual aids and activity programs in elementary and secondary education might there not be some need for it on a collegiate level? Admittedly learning can be more verbal in higher education, but need it be entirely so?

In the courses above will be found the following opportunities for manual skills: Creation in the graphic arts of drawing, painting and sculpture; Participation in individual and group music performance; Production of educational motion pictures; Production of radio programs; Experiments in the science courses; Building of models and museum exhibits for the sciences; Sewing, cooking and decorating exercises in the home arts course; and building, electrical and plumbing exercises in the home repairs course.

### Mathematical Education

*THESIS 8: The program of mathematical education should include a* wide *background in mathematics, rather than an intensive background in a few mathematical fields.*

Hogan and Samuelson (Bib. #14) made an investigation for the Commission on Unit Courses and Curricula of the North Central Association of the Colleges and Secondary Schools in order to determine the mathematics courses most desirable for the training of teachers of secondary mathematics. They questioned 500 principals and teachers from 100 schools. The courses named and the percentage of times each was given is as follows: College Algebra—98%; Plane Trigonometry—97%; Solid Geometry—91%; Analytic Geometry—91%; History of Mathematics—89%; Differential Calculus—82%; Integral Calculus—80%; Theory of equations—65%; Descriptive geometry—49%; and Spherical Trigonometry—46%. Of course, there is no way to tell how many answers were based upon the past mathematics of the person answering without his evaluating the worth of the courses he had taken.

In his book on "The Training of Mathematics Teachers," (Bib. #38) Turner lists the mathematics courses most frequently offered in each of the four years of universities, colleges and teachers colleges in their program for training teachers of mathematics.

*Freshman year:* algebra, trigonometry, analytic geometry, elements of calculus.
*Sophomore year:* differential and integral calculus, analytic geometry.
*Junior year:* analytic geometry (3 dimensions), calculus, differential equations, statistics.
*Senior year:* advanced calculus, theory of equations, projective geometry, statistics, history of mathematics.

In Table II are listed the courses recommended by thirteen selected institutions. Table III is a compilation of seven sets of recommendations of the proper courses for teacher training, together with averages of the recommendations and a new set of courses here recommended. In column #9, which lists the present recommendation, the 25 semester-hours minimum requirement should also be considered the minimum number of credits. If any of the courses listed under Freshman mathematics (College algebra, plane trigonometry, solid geometry or spherical trigonometry) have been used for college entrance credits they should not be repeated for college credit, but substitutions made

TABLE II

*Courses Recommended by Thirteen Selected Institutions for Teachers of High School Mathematics**

| | General Survey | College Algebra | Trigonometry | Plane Analytics | Solid Analytics | Dif. Calculus | Int. Calculus | Adv. Calculus | College Geometry | Professionalized course in Math. | Directed Teaching | Solid Geometry | History of Math. | Differential Equ. | Descriptive or Projective Geom. | Statistics | Astronomy | Theory of Equ. | Physics | Chemistry | Field work in mathematics | Fundamental problems of Geom. | Statics and Dynamics | 6 hours basic Math. Courses |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Brown University | | | √ | √ | | √ | √ | √ | | | | | | √ | | √ | | √ | √ | | | √ | | |
| Duke University | | √ | √ | √ | | √ | √ | | | √ | | | | | | | | √ | | | | | | |
| Fla. State College for Women | | √ | | √ | √ | √ | √ | | √ | | | | | | | √ | | | | | | | | |
| George Peabody College | | √ | √ | √ | | √ | √ | | √ | √ | √ | | | | | | | √ | √ | | | | | |
| N. J. College for Women, New Brunswick | | √ | √ | √ | | √ | √ | | | | | | | | | | | | √ | | | | | √ |
| Teachers College, Columbia Univ. | | | | √ | | √ | √ | | | √ | | | √ | | | | | | | | √ | | | |
| University of Colorado, Boulder | | √ | √ | √ | | √ | √ | √ | | √ | √ | | | √ | | | √ | √ | √ | | | | √ | |
| University of Michigan | | √ | √ | √ | | √ | √ | | | √ | | | √ | | √ | | | √ | | | | | | |
| University of North Carolina | | √ | | √ | | √ | √ | | √ | | | | | √ | | | | √ | | | | | | |
| University of South Carolina | √ | √ | √ | √ | √ | √ | √ | | √ | | | | | √ | | | | √ | | | | | | |
| Woman's College of Texas | | | | √ | | √ | √ | | | √ | √ | | √ | | | | | √ | | √ | | | | |
| Woman's College, Univ. of N. C. | | √ | √ | √ | | √ | √ | | √ | | | √ | √ | | | | | √ | √ | | | | | |
| Winthrop College | | √ | √ | √ | √ | √ | √ | | | | | | | | | | | | | | | | | |

* This table was compiled by Miss Ruth W. Stokes, Syracuse University, Syracuse, New York.

TABLE III

*Mathematics Recommended by Various Reports*

| Report | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Subject | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des |
| Freshman Math. | 6 | 6 | 6 | 6 | 6 | 6 | — | — | — | — | — | — | — | — | 5.0 | 5.4 | 5 | 6 |
| Analytic Geometry | 3 | 3 | 3 | 6 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3.0 | 3.4 | 3 | 3 |
| Calculus | 9 | 9 | 6 | 8 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 3 | 3 | 6.0 | 6.3 | 6 | 6 |
| Projective Geometry | 3 | 3 | — | — | — | 3 | 3[b] | 3[b] | — | — | — | 3 | — | 3 | .9 | 2.1 | — | — |
| College Geometry | — | 3 | 3 | 3 | 3 | 3 | — | — | — | — | — | — | 3 | 3 | 1.3 | 1.7 | — | — |
| Modern Geometry | — | 3 | 3 | 6 | — | — | — | 3 | 3 | 3 | 6 | 6 | — | — | 1.7 | 3.0 | 3 | 6 |
| Fundamental Concepts | 12 | 12 | 3 | 5 | 3 | 3 | — | — | — | — | — | — | — | 3[a] | 2.6 | 3.3 | 3 | 3 |
| Theory of Equations | — | — | — | — | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | — | 3 | 1.7 | 2.1 | — | 3 |
| Higher Algebra | — | — | — | — | — | 3 | — | 3 | — | — | — | — | — | — | — | .9 | 3 | 3 |
| Advanced Calculus | — | — | — | — | — | 3 | — | 3 | — | — | — | — | — | — | — | .9 | — | 3 |
| Complex Variable | — | — | — | — | — | — | — | — | — | — | — | 3 | — | — | — | .4 | — | 3 |
| Theory of Numbers | — | — | — | — | — | — | — | — | — | — | — | 3 | — | — | — | .4 | — | 3 |
| Non-Euclidean Geometry | — | — | — | — | — | — | 3[b] | 3[b] | — | — | — | — | — | 3 | — | .4 | 2 | 3 |
| Differential Equations | — | — | — | — | — | 3 | — | 3 | — | — | — | — | — | — | — | .9 | — | 3 |
| Total | 33 | 39 | 24 | 34 | 24 | 35 | 18 | 27 | 15 | 15 | 20 | 38 | 15 | 30 | 21.3 | 31.3 | 25 | 45 |

| Column | Date | Author | Bib. # |
|---|---|---|---|
| 1 | 1923 | Committee on Mathematical Requirements | 26 |
| 2 | 1934 | Survey of Teacher Education | 28 |
| 3 | 1935 | Commission on Training and Utilization | 9 |
| 4 | 1940 | Joint Commission of MAA and NCTM | 19 |
| 5 | 1940 | Olds | 30 |
| 6 | 1941 | Boston University, CLA, Dept. of Math. | — |
| 7 | 1946 | Commission on Post-War Plans | 8 |
| 8 | — | Average of Columns 1–7 | — |
| 9 | — | Present suggested program | — |

[a] Included in courses in College algebra and geometry.
[b] Projective geometry and non-Euclidean geometry combined.

TABLE IV

*Courses in Fields Related to Mathematics Recommended in Various Reports*

| Report | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Subject | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des | Min | Des |
| Statistics | — | 3 | 3 | 6 | 3 | 3 | 3[c] | 3[c] | 3 | 3 | 3 | 3 | — | 3 | 2.1 | 3.4 | 3 | 3 |
| Economics | — | 3 | — | — | 3 | 3 | — | — | 3 | 3 | — | — | 3 | 3 | 1.3 | 1.7 | — | — |
| Mathematics of Finance | — | 3 | — | — | 3 | 3 | (3)[c] | (3)[c] | — | — | — | — | 3 | 6 | .9 | 1.7 | — | 3 |
| Physics | 9 | 9 | — | — | 6 | 9 | 6[d] | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 5.6 | 6.0 | 6 | 9 |
| Other Physical Science | 6 | 12 | — | — | 6 | 12 | (6)[d] | (6)[e] | — | — | 6 | 12 | — | — | 2.6 | 6.0 | 3 | 6 |
| Astronomy | — | 3 | — | — | — | 3 | (6)[d] | (6)[e] | — | — | — | — | 3 | 3 | .4 | 1.3 | — | 3 |
| History of Mathematics | — | — | 2 | 3 | — | — | 3 | 3 | 3 | 3 | 3 | 3 | — | 3 | 1.6 | 2.1 | 3 | 3 |
| Math. in Modern Life | — | — | 3 | 3 | — | — | — | — | 3 | 3 | — | — | — | — | .9 | .9 | — | — |
| Use of Instruments | — | — | — | — | — | — | — | — | — | — | — | — | — | 3 | — | .4 | — | 3 |
| Mechanics | — | — | — | — | — | — | — | — | 3 | 3 | — | 3 | 3 | 3 | .9 | 1.3 | — | — |
| Navigation | — | — | — | — | — | — | — | — | — | — | — | — | 3 | 3 | .4 | .4 | — | — |
| Total | 15 | 33 | 8 | 12 | 21 | 33 | 12 | 18 | 21 | 21 | 18 | 27 | 21 | 33 | 20.0 | 25.2 | 15 | 30 |

*Note*: Meanings of columns same as in Table III.
[c] Statistics and mathematics of finance combined.
[d] An introductory course of Physics, astronomy or chemistry that makes some use of mathematics.
[e] At least one more of the three sciences of physics, chemistry and astronomy.

from the other courses desirable to complete the 25 hours. More mathematics courses above the basic 25 hours should be allowed as electives, but not encouraged.

*THESIS 9: Pure mathematics must be buttressed by courses in related fields.*

Although the ideal situation would be to have many illustrations of the uses of mathematics and applications to other fields in the courses in pure mathematics, such a procedure is not always feasible on a college level. It destroys the continuity of the mathematical development. It is also impossible to give enough time to illustrative uses of mathematics to meet the needs and interests of all the types of pupils in the classes. And yet such examples must be constantly used in secondary school mathematics classes to enliven and enrich the pure mathematics, for on that level teaching is much less abstract. Therefore additional courses in applications of mathematics must be included.

The recommendations summarized in Table IV show the courses that have been mentioned in various reports.

*THESIS 10: A background of paid experience in some particular field in which mathematics is used is extremely desirable.*

Mathematics, being abstract, can exist and grow without any great need for its applications, but it is these applications which give direction and inspiration to the development and learning of mathematics. Theoretical courses in methods by which mathematics can be applied, talks by practicing mathematicians, and visits to industrial situations which rely on mathematics are inspirational, but nothing gives the prospective teacher the respect for his subject and the right to parade it in the company of subjects usually called more "practical" than the experience of earning money by mathematics, other than by teaching it.

There is another benefit to be gained from contact with the business world: the opportunity to see and feel the attitudes of people who are not interested in an academic atmosphere. Few of the pupils

whom teachers are supposed to influence will spend their lives in schools; how can teachers who have passed from one library to another honestly appreciate the problems for which they prepare their students?

The Commission on Post-War Plans (Bib. #8, p. 129) advises that teachers "devote at least a summer or two in learning a variety of jobs in one of the large manufacturing plants." There is certainly an advantage in seeing a variety of applications if enough time is spent at each job to assure that the teacher knows the job and has not merely learned of the existence of such an application. However, there is no reason why such experiences should be limited to manufacturing plants; professional offices, public utility concerns, insurance companies, banks, the distributive occupations (retail and wholesale concerns), service occupations (hotels, restaurants, etc.) and even agriculture, fishing and mining have some possibilities. Any of this work which is approved and supervised by the college should certainly carry credit toward a degree.

## PROFESSIONAL EDUCATION

*THESIS 11: The required courses in education should be organized around major questions in education, of comparable types of material, and of complementary, non-overlapping subjects.*

The majority of the courses in education today are unsuitable to be formed into combinations which will give a complete and economical background. This is due primarily to the state requirements for certification which have frozen titles on courses and made it impossible to revise the courses as conditions and knowledge change. On the secondary level courses are being shortened, combined, lengthened and revised to meet the objectives of education as they are being more carefully defined. The same procedure should be allowed on the college level. (See Thesis #2, above).

The first step in designing new courses which will cover the field of education is to find the major areas or questions which should be covered. These should be as broad as possible so that there will be only a few foci of attention about which to organize courses, and yet not so broad that they are vague and philosophical. These areas should be chosen as much as possible at the same level of generality so that a course in "The Meaning of Education" is not combined with "The Production of Educational Motion Pictures." Traditional administrative requirements would probably require similar lengths of time for these two units, and they surely are not of equal importance. The final consideration is the elimination of overlapping. Much justified criticism has been directed at education for the amount of material which appears in many, different courses. There should be conscious efforts to prevent this wastage.

In order to determine the professional courses which have been thought worthwhile in the past let us summarize many of the past recommendations. They will be found in Table V.

All recommendations in Table V were stated definitely as separate courses except those listed from Butler and Wren. They stated areas that needed attention, and putting them into the table as semester-hours may very well be misinterpreting their meaning.

Many of the courses listed in Table V are not suitable for an undergraduate program: "Introduction to Educational Concepts" is so broad that it will overlap most of the other courses offered; "General Psychology" in addition to "Educational Psychology" would surely cause duplication; "School Management" is too small a topic to be allowed a course by itself; while "Research and Experiment" is not so important to a beginning teacher as the other courses which should replace it.

Schorling (Bib. #35, pg. 196) at the Institute for Administrative Officers of Higher Institutions, tried to collect the major areas to be considered in the train-

TABLE V

*Professional Courses Suggested*

| Source | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Course | | | | | | | | | |
| General Psychology | 3 | 3 | 3 | — | 3 | 3 | 3 | 3 | 2.6 |
| General Method | 3 | 3 | — | 3 | 3 | — | 3 | 3 | 2.3 |
| School Management | 3 | — | — | — | — | — | — | — | .4 |
| Teaching of Mathematics | 3 | 3 | 6 | 3 | — | 3 | 6 | 3 | 3.4 |
| History of Education | — | 3 | — | — | — | — | — | — | .4 |
| Principles of Education | — | 3 | 3 | 2 | — | — | — | — | 1.0 |
| Educational Psychology | — | 3 | — | — | — | — | 3 | — | .8 |
| Organization and function of secondary education | — | 3 | — | 2 | 3 | — | 3 | 3 | 1.8 |
| Introduction to education concepts | — | — | 3 | — | — | 3 | — | — | .8 |
| Measurement | — | — | 3 | 2 | 2 | — | 3 | 3 | 1.6 |
| Pupil-Guidance | — | — | — | — | — | — | — | 3 | .4 |
| Research and Experiment | — | — | — | — | — | — | — | 3 | .4 |
| Total | 12 | 21 | 18 | 12 | 11 | 9 | 21 | 21 | 15.6 |

| Column | Date | Author | Bib. # |
|---|---|---|---|
| 1 | 1920 | Learned | 24 |
| 2 | 1923 | Committee on Mathematical Requirements | 26 |
| 3 | 1934 | Survey of Teacher Education | 28 |
| 4 | 1935 | Commission on Training and Utilization | 9 |
| 5 | 1940 | Joint Commission of MAA and NCTM | 19 |
| 6 | 1940 | Olds | 30 |
| 7 | 1941 | Boston University, CLA, Dept. of Math. | — |
| 8 | 1941 | Butler and Wren | 5 |
| 9 | — | Average of Columns 1–8 | — |

ing of teachers into five groups:

1. Training in the organization of teachable units of work so as to adjust the curriculum to the ever-changing needs of the school and the community.

2. Training in the philosophy and history of education, and in the techniques of educational research, so as to provide a scale of values for placing the proposals and the practices of the teaching art in their true and proper perspective.

3. Training in educational psychology and in the special methods of one or more subjects, so as to keep one's practice up to date, to make possible, in fact, a true appraisal of results.

4. Wide experience in directed teaching to provide a basis for such reading and thinking as may be necessary for a clear and intelligent understanding of pedagogical principles.

5. Training in case work according to sound clinical procedures and in the compilation and use of records in order to facilitate perhaps the most important job of the school, guidance in the integration of the individual pupil's personality.

Schorling's areas are noteworthy for trying to organize the professional needs of teachers-in-training before setting up courses to meet these objectives. This is exactly the approach recommended in this article. There are a few objections to this treatment: they contain more than five distinct areas of emphasis, they are a mixture of material to be taught and objectives of teaching, and they are not of the same level of generality.

It is suggested that 30 semester-hours be allowed for professional education (See Thesis #3, above) and that it be divided as shown below. A course of six semester hours is assumed to run for two semesters; three semester-hours, for one semester. (Except Directed Teaching, for which see Thesis #13, below). After each course suggested appear the names of traditional courses which would contribute their material to the new course. It is recognized that many toes will be trod upon by minimizing the time devoted to the pet subjects of certain individuals, but the time allotment is based upon a revaluation of the needs of teachers-in-training. (See Thesis #2, above).

*Education and Society* 6 semester-hours
History of Education, School and Society, Educational Sociology, Intercultural Education.
*Characteristics of Learning* 3 semester-hours
Educational Psychology, General Psychology, Psychology of Learning.
*Secondary Education* 3 semester-hours
Principles of Secondary Education, Issues of Secondary Education, Principles of Guidance.
*The Teacher-Learner Situation* 6 semester-hours
Methods of Teaching in the Secondary School, Unit Method in the Secondary School, Audio-visual aids to Learning.
*Teaching of Mathematics* 3 semester-hours
Methods and Curriculum of Secondary Mathematics
*Educational Measurements* 3 semester-hours
*Directed Teaching* 6 semester-hours
(See Thesis #13, below, for directed teaching)

*THESIS 12: More attention should be devoted to the developing of essential personal qualities in teachers.*

Most of the education of teachers is directed toward improving their intellectual background, in mathematics and general cultural subjects, and in giving them a technical excellence in the trade of teaching. And yet, if teachers are going to be citizens who teach by their contributions and examples to the community there must be some provision for consciously training these essentially personal qualities. (See Thesis #4, above).

Noyer (Bib. #29, pg. 210–212) has outlined the qualities to be developed. Examples and fuller descriptions will be found in the original article. Here is his list:

Desirable Social Qualities: Consideration for others, Ability to enlist good will, Reliability, Good citizenship, High Moral Standards.
Diversified Interests: The fine arts, People, Present-day Problems, Appreciation of nature, Recreations.
Desirable Tempermental Traits: Happy disposition, Responsiveness, Self-control, Firmness.
Pleasing Personal Appearance and Habits: Engaging manner, well-groomed appearance, Poise, Good wholesome physique, Pleasant speaking voice.
Commendable Work Habits and Attitudes: Industriousness, Sense of responsibility, Love of work, Initiative, Punctuality, Neatness and Orderliness.
Good Health and Physical Efficiency: Good physical health, Mental health, Absence of any outstanding physical defect which would interfere with teaching.

The exact methods to be used in teaching and testing these qualities is still one of the major issues in teacher training. Many of them can be planned for in the courses in general education (See Thesis #5 & 6, above). It will be possible to detect and guide the development of them in the work in directed teaching. (See Thesis #13, below).

*THESIS 13: Opportunity for observation of competent teaching and for participation in directed teaching must be part of the program.*

The work in directed teaching is usually cited by teachers as the most valuable part of their teacher training. This should not be regarded as an indication that the greater part of the professional training should be in observation and directed teaching; there are too many other aspects of the training.

The Joint Commission (Bib. #18, p. 191) stresses the importance of this work:

> The most important element in professional training is student practice teaching, carried out under the most competent supervision that can be procured. The Commission considers this work so important that it urges even greater attention to be paid to it in the future than in the past. Work in practice teaching should be preceded by a good course in methods; and in this course special stress should be put upon the topics that are most intimately connected with the ideas, the concepts, and the basic processes of mathematics. Some phases of the work of the methods teacher should be carefully correlated with instruction that students receive in mathematics.

Other references to the great desirability of competent directed teaching will be found in the following books and articles in the bibliography: #9, 19, 24, 26, 28, 30 and 34. The National Committee on Mathematical Requirements (Bib. #26) asks for the satisfactory performance of the duties of a teacher of mathematics in a secondary school for a period of not less than 1 year, or 20 semester hours. This substitution of actual teaching for the directed teaching requirement is usual and proper.

If six-semester hours of credit are al-

lowed for directed teaching (See Thesis #11, above) it may be possible to require a full month of attendance at the school, throughout the school day to make participation in extra-curricula and planning activities possible.

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# The Geometry of the Pentagon and the Golden Section

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The Geometry of the Pentagon has become almost a foster-child besides other chapters of geometry, as for instance the geometry of the triangles or of the quadrilaterals. Considering terminologies, we find the whole field of trigometry deriving its name from the geometry of triangles and the "quadrature of areas" (quadratum = square) from the regular representative of the quadrilaterals, all units for measuring areas being also squares.

The characteristic elements of the geometry of the pentagon are neither related to the trigonometric reproduction of forms nor to measuring areas. The regular pentagon, however, and especially the regular stellar pentagon formed by its diagonals, the pentagram, are used today in the flags and emblems of the mightiest nations and had a similar use already two and a half thousand years ago when the pentagram was the emblem of the Pythagorian School. It is the particular appeal of the pentagon to the sense of beauty, and the unique variety of mathematical relationships connected with it which are the characteristics of the geometry of the pentagon. This geometry is therefore particularly fit to stimulate mathematical interest and investigations. Outstanding among the mathematical facts connected with the pentagon are the manifold implications of the irrational ratio of The Golden Section.

The first figure shows a regular pentagon, and inscribed in it the pentagram formed by its diagonals. The central area of the pentagram forms again a regular pentagon in reverse position. In this pentagon another pentagram has been inscribed. The total diagram of Figure 1 contains three horizontal lines, among them the base of the pentagon. Due to symmetry there is a group of three parallel lines coordinated in the same way to every one of the five sides of the pentagon. These parallel lines form between them two types of rhombi, smaller and larger ones. One of

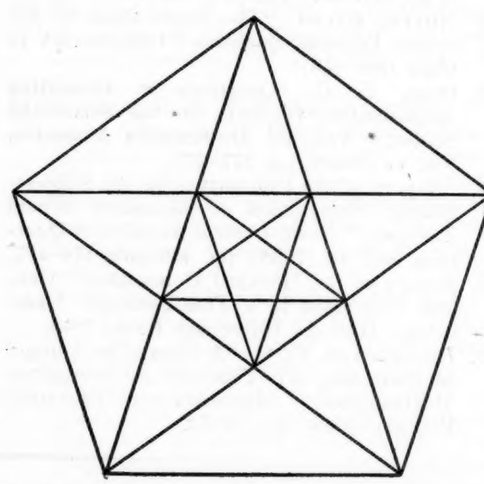

**Fig. 1. Pentagon with inscribed pentagrams.**

the smaller and one of the larger rhombi is marked in Figure 2 and Figure 3. A diagonal divides a rhombus into two congruent isosceles triangles. By folding and bending over the marked rhombus in Figure 2 along its horizontal diagonal we shall always reach exactly the opposite vertex of the central area. By cutting a pentagram out of paper, then bending over its outer parts and holding the paper before a light will make the inner pentagram appear in the central area. Folding the marked rhombus of Figure 3 in the same way along its horizontal diagonal will bring on both ends of this diagonal two angles to coincidence into which the diagonals divide the interior angles of a regular pentagon. Consequently, a pentagram trisects the interior angles of a circumscribed pentagon. If one of the partial angles is denoted $\phi$ the angles of the large rhombus of Figure 3 are $2\phi$; $3\phi$; $2\phi$; $3\phi$ and

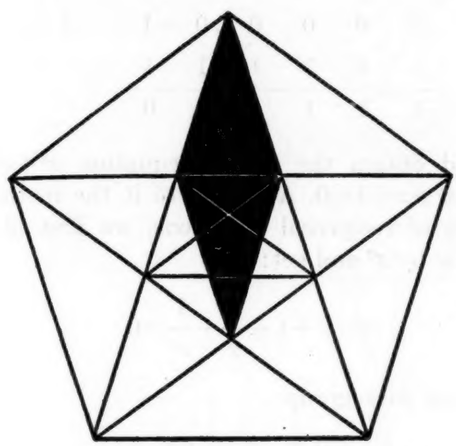

**FIG. 2. Smaller rhombus contained in the pentagon-pentagram diagram.**

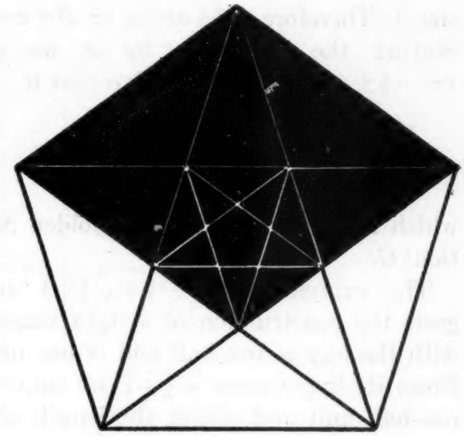

**FIG. 3. Larger rhombus contained in the pentagon-pentagram diagram.**

those of the small rhombus in Figure 2: $\phi$; $4\phi$; $\phi$; $4\phi$. The sum of the angles of any of the two rhombi being $10\phi$ $\phi = 360°/10 = 36°$. All angles which come in the diagrams of the Figures 1–3 are of the sizes: 36°; 72°; 108°; 144°; 180°; 216°; 252°; 288°; 324°; and 360°, thus forming an arithemetic progression with a difference of 36°.

The line segments in Figure 1, including both the partial segments between points of intersection and also their sums, are of six different sizes. The largest are the diagonals of the large pentagon. Counting them as of size No. 1 and then continuing with numbering until we come to size No. 6 with the sides of the inmost pentagram, the various sizes appear in the following quantities:

contained which are smaller and have sides of the sizes 5 and 6. The similarity of all these triangles establishes the following equations of the ratios of the line segments:

$$\frac{\text{segm } 1}{\text{segm } 2} = \frac{\text{segm } 2}{\text{segm } 3} = \frac{\text{segm } 3}{\text{segm } 4}$$

$$= \frac{\text{segm } 4}{\text{segm } 5} = \frac{\text{segm } 5}{\text{segm } 6}.$$

Therefore, the six sizes of line segments are members of a geometric progression. Whereas the angles in the pentagon-diagram make up an arithmetic progression the line segments form a geometric progression. Denoting $x$ as the ratio of this geometric progression and "$a$" for the length of a line segment of size 1, we have:

Line segments of size No. 1 come in the diagram 5 times:
Line segments of size No. 2 come in the diagram 15 times:
Line segments of size No. 3 come in the diagram 15 times:
Line segments of size No. 4 come in the diagram 15 times:
Line segments of size No. 5 come in the diagram 10 times:
Line segments of size No. 6 come in the diagram 5 times:

Total amount of line segments 65

In Figure 4, three isosceles triangles which are contained in the pentagram are marked through shading. The sides of the largest one are of the sizes 1; 1; 2. The sides of the middle sized triangle are: 2; 2; 3 and those of the smallest triangle: 3; 3; 4. In the complete diagram of Figure 1 further triangles of still the same form are

segment size No. 1 $= a$
segment size No. 2 $= ax$
segment size No. 3 $= ax^2$
segment size No. 4 $= ax^3$
. . . . . . . . . . . . . . . . . . . . . . . .
segment size No. $n$ $= ax^{n-1}$

The value of $x$ can be found through the fact that one line segment of size 3 and one of size 2 make up a pentagram side of

size 1. Therefore $ax^2+ax=a$ or $x^2+x=1$. Solving the quadratic for $x$ we get $x=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+1}$. The positive root is

$$-\frac{1}{2}+\frac{\sqrt{5}}{2}=\frac{\sqrt{5}-1}{2}=0.61803398875\cdots$$

which is the number of the Golden Section: $G$.

The expression $G=-\frac{1}{2}+\sqrt{\frac{1}{4}+1}$ suggests the construction of a right triangle with the legs of one-half and of one unit. From its hypotenuse $\sqrt{\frac{1}{4}+1}$ we subtract one-half unit and obtain the length of $G$

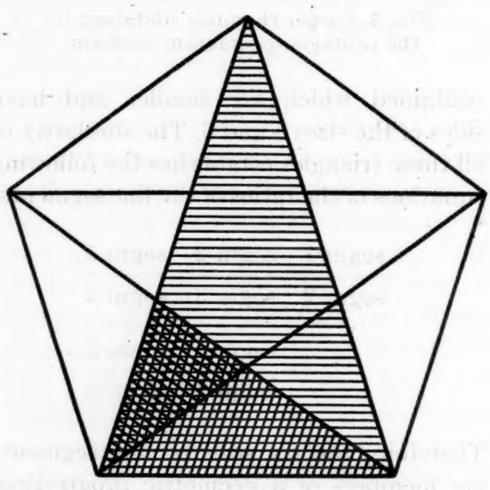

FIG. 4. Similar triangles in a pentagram.

units. Starting the construction with any given line segment, one obtains the original length multiplied by the factor $G$. All the 65 line segments of the diagram in Figure 1 can thus be obtained from the large pentagram side by repeated application of the described construction.

Other lengths connected with the pentagram, for instance, the relative altitudes of its vertices can also be expressed through $G$. This can be done by applying the theory of complex roots of an equation and of the complex-number plane. The geometry of a regular $n$ sided polygon reappears in the $n$th roots of unity. For the pentagon, we use a 5th root of unity corresponding to the equation $x^5-1=0$. One of the roots being 1, we get through synthetic division:

$$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & \end{array}$$

and obtain the quartic equation $x^4+x^3+x^2+x+1=0$. Applying to it the methods of reciprocal equations, we first divide by $x^2$ and get:

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0.$$

Then we regroup:

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0.$$

Substituting $y$ for $x+(1/x)$ and therefore $y^2$ for $x^2+2+(1/x^2)$ or $y^2-2$ for $x^2+(1/x^2)$ the equation takes on the form $y^2+y=1$ which is again the characteristic equation which has $G$ as its positive root. The two roots $y=\frac{-1\pm\sqrt{5}}{2}$ can be expressed through $G$ as $\frac{-1+\sqrt{5}}{2}=G$ and $\frac{-1-\sqrt{5}}{2}=-(G+1)$. The values for $x$ are obtained by solving the equations:

$$x+\frac{1}{x}=\frac{-1+\sqrt{5}}{2}=G \quad \text{and}$$

$$x+\frac{1}{x}=\frac{-1-\sqrt{5}}{2}=-(1+G)$$

By multiplying the first equation with $x$ we get: $x^2-Gx=-1$. Its roots are:

$$x=\frac{G}{2}\pm\sqrt{\frac{G^2}{4}-1}.$$

By multiplying the second equation with $x$ we get: $x^2+(1+G)x=-1$ and the roots are

$$x=-\frac{1+G}{2}\pm\sqrt{\frac{(1+G)^2}{4}-1}.$$

As $G<1$ both $G^2/4$ and $(1+G)^2/4$ are smaller than 1 and consequently all the four roots are complex. The five roots of the original equation $x^5-1=O$ are:

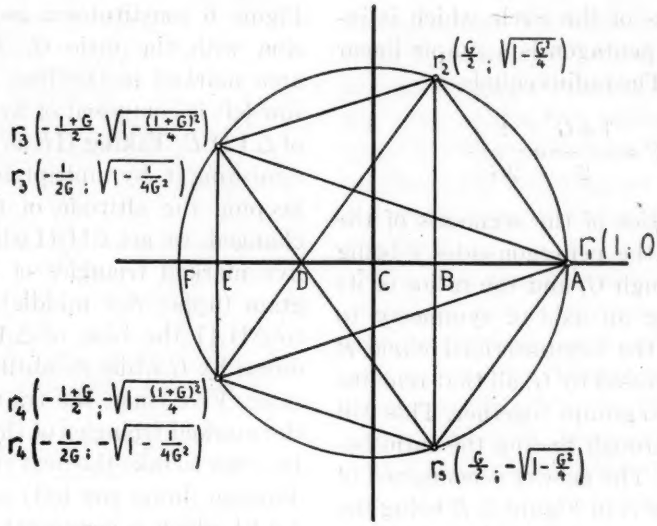


**FIG. 5. The ratio $G$ in the roots of the equation: $x^5-1=0$.**

$$r_1= \quad 1$$

$$r_2= \quad \frac{G}{2}+\sqrt{1-\frac{G^2}{4}}\cdot i$$

$$r_3=-\frac{1+G}{2}+\sqrt{1-\frac{(1+G)^2}{4}}\cdot i$$

$$r_4=-\frac{1+G}{2}-\sqrt{1-\frac{(1+G)^2}{4}}\cdot i$$

$$r_5= \quad \frac{G}{2}-\sqrt{1-\frac{G^2}{4}}\cdot i$$

**The expressions for $r_3$ and $r_4$ can be simplified through the relation $1+G=1/G$ which derives itself from the fundamental equation $x^2+x=1$ through dividing it by $x$: $x+1=1/x$. As $G$ is its positive root we have $G+1=1/G$. Therefore:**

$$r_3=-\frac{1}{2G}+\sqrt{1-\frac{1}{4G^2}}\cdot i;$$

$$r_4=-\frac{1}{2G}-\sqrt{1-\frac{1}{4G^2}}\cdot i.$$

**The Figure 5 shows the location of the five roots on the complex number plane. They lie on the circle with the radius of one unit. The abscissae of the five points are the real parts of the roots:**

$$1;\quad \frac{G}{2};\quad -\frac{1+G}{2}=-\frac{1}{2G};$$

$$-\frac{1+G}{2}=-\frac{1}{2G};\quad \frac{G}{2}$$

**and the ordinates the imaginary parts:**

$$0;\quad \sqrt{1-\frac{G^2}{4}};$$

$$\sqrt{1-\frac{(1+G)^2}{4}}=\sqrt{1-\frac{1}{4G^2}};$$

$$-\sqrt{1-\frac{(1+G)^2}{4}}=-\sqrt{1-\frac{1}{4G^2}};$$

$$-\sqrt{1-\frac{G^2}{4}}.$$

**From the abscissae we obtain the ratios of the line segments in a pentagram along its axis of symmetry to the radius of the circumscribed circle. They are all simple linear functions of $G$.**

$$AB=1-\frac{G}{2}$$

$$BC=\frac{G}{2}$$

$$CD=BD-BC=AB-BC$$

$$=1-\frac{G}{2}-\frac{G}{2}=1-G$$

$$DE=CE-CD=\frac{1+G}{2}-(1-G)=\frac{3G}{2}-\frac{1}{2}$$

$$EF=1-\frac{1+G}{2}=\frac{1}{2}-\frac{G}{2}=\frac{1-G}{2}.$$

Also the radius of the circle which is inscribed in the pentagon is a simple linear function of $G$. The radius equals

$$EC=\frac{1+G}{2}=\frac{1}{2G}.$$

Thus the ratios of the segments of the pentagram to the pentagon-sides $s$ being expressed through $G$, and the ratios of its segments along an axis of symmetry to the radius of the circumscribed circle $R$ being also expressed by $G$, all that remains is to tie the two groups together. This will be achieved through finding the ratio between $s$ and $R$. The answer is contained in the ordinate for $r_3$ in Figure 5. $R$ being the radius of the circumscribed circle, half the side of the pentagon is

$$\frac{s}{2}=\sqrt{1-\frac{1}{4G^2}}\cdot R \quad \text{or} \quad \frac{s}{R}=2\sqrt{1-\frac{1}{4G^2}}$$

which again expresses itself through $G$.

The number $G$ is also the ratio of areas which are formed between the pentagon and the pentagrams. The sequence of areas which is marked in the five diagrams of Figure 6 constitutes a geometric progression with the ratio $G$. The ring-shaped area marked in the first diagram (upper row left is composed of five times the area of $\triangle$ I II $E$. Taking II$E$ as the base and diminishing it by multiplying with $G$ while keeping the altitude of the triangle unchanged, we get $\triangle$II$A$I which is one of the five marked triangles of the second diagram (upper row middle). In comparison to $\triangle$II$A$I the base of $\triangle AE$I is again reduced by $G$ while its alititude remains the same. Five times the triangle $AE$I equals the marked triangles of the third diagram. In order to take the next step to the fourth diagram (lower row left) we consider again $\triangle AE$I which is congruent to $\triangle AEC$. Taking $AC$ as its base and reducing it by the ratio $G$ to $CH$ without changing the altitude we get $\triangle HCE$ which is congruent to $\triangle CED$. By subtracting from the triangle $CED$ the triangle $EDK$ and adding instead the congruent triangle $CDF$ we obtain the quadrilateral $CKDF$ which taken five times makes up the marked area of the fourth diagram. This area, therefore, represents the third diagram's area reduced

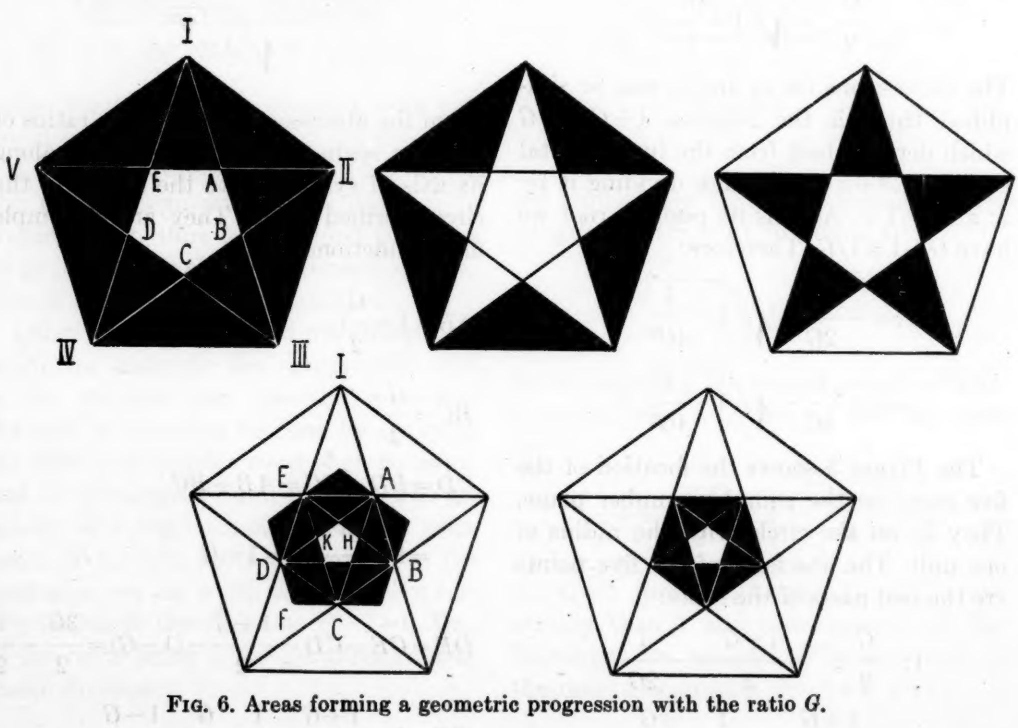


**Fig. 6. Areas forming a geometric progression with the ratio $G$.**

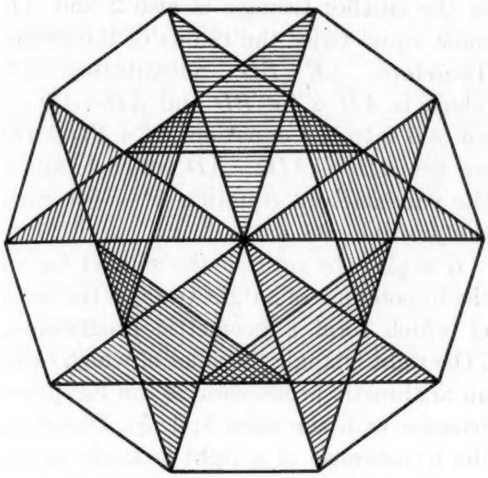

FIG. 7. The ratio $G$ in the regular decagon.

by $G$. Finally, we take up once more the triangle $CED$ which equals one-fifth of the marked area of the fourth diagram. Reducing its base $CE$ by the ratio $G$ without changing the altitude, we obtain the triangle $CKD$ which taken five times makes up the marked ring-shaped area of the fifth diagram. Denoting the total marked area of the first diagram with $A$ the marked areas of the successive diagrams form the geometric progression: $A$; $AG$; $AG^2$; $AG^3$; $AG^4$. The last ring-shaped area occupies the same place within the inner pentagram as the first ring-shaped area in the outer one. The ratio between the two rings is therefore $G^4$ which checks with a previously found result that corresponding sides of the two pentagons have the ratio $G^2$. The white area left over in the middle of the last diagram is also $G^4$ times the white area in the middle of the first diagram.

The part the ratio $G$ plays in a pentagram also carries over into the domain of the regular decagon: $G$ is the ratio of the side of a regular decagon to the radius of its circumscribed circle. Figure 7 shows a regular decagon. Its vertices are joined with the center and thus the angle of 360° around the center is divided into ten equal angles of 36°. Ten pentagrams can be placed around the center to fit in these spaces. Every second of them is drawn in Figure 7 and marked through shading. The sides of these pentagrams equal the radius of the circle circumscribed about the decagon, and the sides of the pentagons drawn around these pentagrams equal the decagon side. The ratio between these two sizes is $G$. The usual construction of the side of a regular decagon to be inscribed in a given circle is an application of $G$.

In solid geometry $G$ reappears in the geometry of the pentagondodecahedron and of the icosahedron which contain

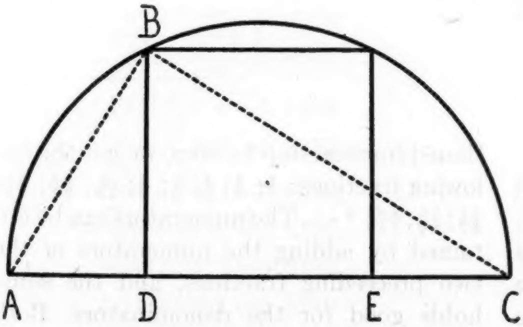


FIG. 8. The ratio $G$ in a square inscribed in a semi-circle.

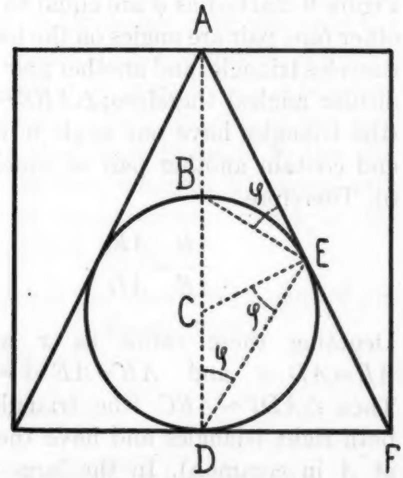


FIG. 9. The ratio $G$ in a circle inscribed in an isosceles triangle which is in turn inscribed in a square.

pentagons as their faces or as plane sections.

The ratio $G$ appears furthermore in geometric figures which are not connected with pentagons or decagons. One of them is a square which is inscribed in a semicircle (Figure 8). Whereas the three line segments of a pentagram side have the smaller one in the middle and the larger ones to the sides, we have the reverse sequence in Figure 8. Nevertheless, the ratio between the two sizes of segments is again $G$. To prove it we use the similar triangles $BCD$ and $ABD$. Denoting the ratio of the shorter to the longer legs as $x$ we have:

$$\frac{BD}{CD}=x; \quad \frac{AD}{BD}=x$$

and therefore $BD=CD\cdot x$; $AD=BD\cdot x=CD\cdot x^2$. As $AD+BD=AD+DE=AE$ and $AE=CD$ we have: $CDx^2+CDx=CD$ or $x^2+x=1$, the positive root being $G$. This result can also be interpreted for solid geometry, dealing with an equilateral cylinder inscribed in a hemisphere.

Another appearance of $G$ occurs in a circle inscribed in an isosceles triangle which in turn is inscribed in a square (Figure 9) or, interpreted by solid geometry, in a sphere inscribed in a cone which in turn is inscribed in an equilateral cylinder or in a cube. The three angles in Figure 9 marked as $\phi$ are equal to one another (one pair are angles on the base of an isosceles triangle, and another pair perpendicular angles) therefore; $\Delta ABE \sim \Delta AED$ (the triangles have one angle in common and contain another pair of equal angles $\phi$). Therefore

$$\frac{AB}{AE}=\frac{AE}{AD}.$$

Denoting these ratios as $x$ we have $AE=AD\cdot x$ and $AB=AE\cdot x=AD\cdot x^2$. Then $\Delta ADF \sim AEC$ (the triangles have both right triangles and have their angle at $A$ in common). In the large triangle $ADF$ the ratio of the larger to the smaller leg is 2; therefore the corresponding ratio in the smaller triangle is also 2 and $AE$ must equal twice the radius of the circle. Therefore, $AE=BD$. Substituting $AE$ which is $AD\cdot x$ for $BD$ and $AB=AD\cdot x^2$ for $AB$ into the equation $AB+BD=AD$ we get $ADx^2+ADx=AD$ which is again the fundamental equation $x^2+x=1$ with the positive root $x=G$.

$G$ is also the ratio of the smaller leg to the hypotenuse of a right triangle the sides of which form a geometric progression. (The right triangle the sides of which form an arithmetic progression is the Egyptian triangle with the sides 3; 4; 5). Denoting the hypotenuse of a right triangle whose sides form a geometric progression as "$a$" the larger leg is $a\cdot x$ and the smaller leg $a\cdot x^2$. From the theorem of Pythagoras we get $a^2=(ax)^2+(ax^2)^2$ or $x^4+x^2=1$ which gives for $x^2$ the positive root $G$. Therefore the smaller leg of the right triangle being $a\cdot x^2$ equals $a\cdot G$.

Arithmetically the number $G$ shows also outstanding qualities. First, it has the same infinite sequence of decimals as its reciprocal value: $G=0.61803398875\cdots$. $1/G=1.61803398875\cdots$. It is the only positive number which forms its reciprocal value by adding 1. This results from the equation $x^2+x=1$ by dividing it by $x$: $x+1=(1/x)$. Then $G$ can be expressed as the limit of a continued fraction written only by figures 1. By computing this continued fraction step by step, we get the following fractions:

$$G=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$$

$1$; $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{5}$; $\frac{5}{8}$; $\frac{8}{13}$; $\frac{13}{21}$; $\frac{21}{34}$; $\frac{34}{55}$; $\frac{55}{89}$; $\frac{89}{144}$; $\cdots$. The numerators can be obtained by adding the numerators of the two preceeding fractions, and the same holds good for the denominators. Both the numerators and the denominators form a Series of Fibonacci:

1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; · · ·. in which each term is the sum of the two preceding ones. $G$ is the limit of the ratios of two successive terms in the Series of Fibonacci. This Series starts with two terms of 1. If instead any other numbers are chosen (excluding zero) which can be intigers or fractions and the same procedure is applied to them $G$ will still appear as the limit of the ratios of two successive terms. This is shown in the following example in which arbitrarily the numbers 5 and 24 have been chosen:

$$
\begin{aligned}
&= 5\\
&= 24\\
5+24 &= 29\\
24+29 &= 53\\
29+53 &= 82\\
53+82 &= 135\\
82+135 &= 217\\
135+217 &= 352\\
217+352 &= 569\\
352+569 &= 921\\
569+921 &= 1490\\
921+1490 &= 2411\\
1490+2411 &= 3901\\
\cdots &\; \cdots
\end{aligned}
$$

$$
\begin{aligned}
5\div 24 &= 0.2083\cdots\\
24\div 29 &= 0.8276\cdots\\
29\div 53 &= 0.5472\cdots\\
53\div 82 &= 0.6463\cdots\\
82\div 135 &= 0.6074\cdots\\
135\div 217 &= 0.6221\cdots\\
217\div 352 &= 0.6165\cdots\\
352\div 569 &= 0.6187\cdots\\
569\div 921 &= 0.6178\cdots\\
921\div 1490 &= 0.6181\cdots\\
1490\div 2411 &= 0.6180\cdots\\
2411\div 3901 &= 0.6180\cdots\\
\cdots &\; \cdots
\end{aligned}
$$

In our case, the first four decimals of $G$ are obtained at the 11th division.

$G$ can also be expressed as a limit of square roots in which 1 is again the only figure used:

$$G=\cfrac{1}{\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}}$$

(the proofs which follow the theory of limits are here omitted).

In the history of mathematics references to the number $G$ lead back to oldest geometric records. There is a passage in Herodotus in which he relates that the Egyptian priests had told him that the proportions of the Great Pyramid at Gizeh were so chosen that the area of a square whose sides is the height of the Great Pyramid equals the area of a face triangle. Writing $2b$ for the side of the base of the Great Pyramid (see Figure 10) and "$a$" for the altitude of a face triangle and "$h$" for the height of the Pyramid, Herodotus relation is expressed in the following equation: $h^2=(2b\cdot a)/2=a\cdot b$. As "$a$" is the hypotenuse of a right triangle with the legs "$b$" and "$h$" we can apply the theorem of Pythagoras and get: $a^2=b^2+h^2$ or $h^2=a^2-b^2$. Equating the expressions for $h^2$ in the two equations we obtain $a^2-b^2=ab$ or $b^2+ab=a^2$. Dividing the equation by $a^2$ we have $(b/a^2)+(b/a)=1$. Substituting $x$ for the ratio $b/a$ we are back at the equation $x^2+x=1$ which has $G$ as its positive root. Therefore, $G$ is the ratio of half the side of the base square of the Great Pyramid to the altitude of the face triangle. Checking with the actual measurements taken at the Great Pyramid, we have:

$h=148.2\ m$ (reconstructed height of undamaged apex)

$b=116.4\ m$

which makes

$$a=\sqrt{148.2^2+116.4^2}=188.4$$

and gives the ratio $b/a=0.6178\cdots$.

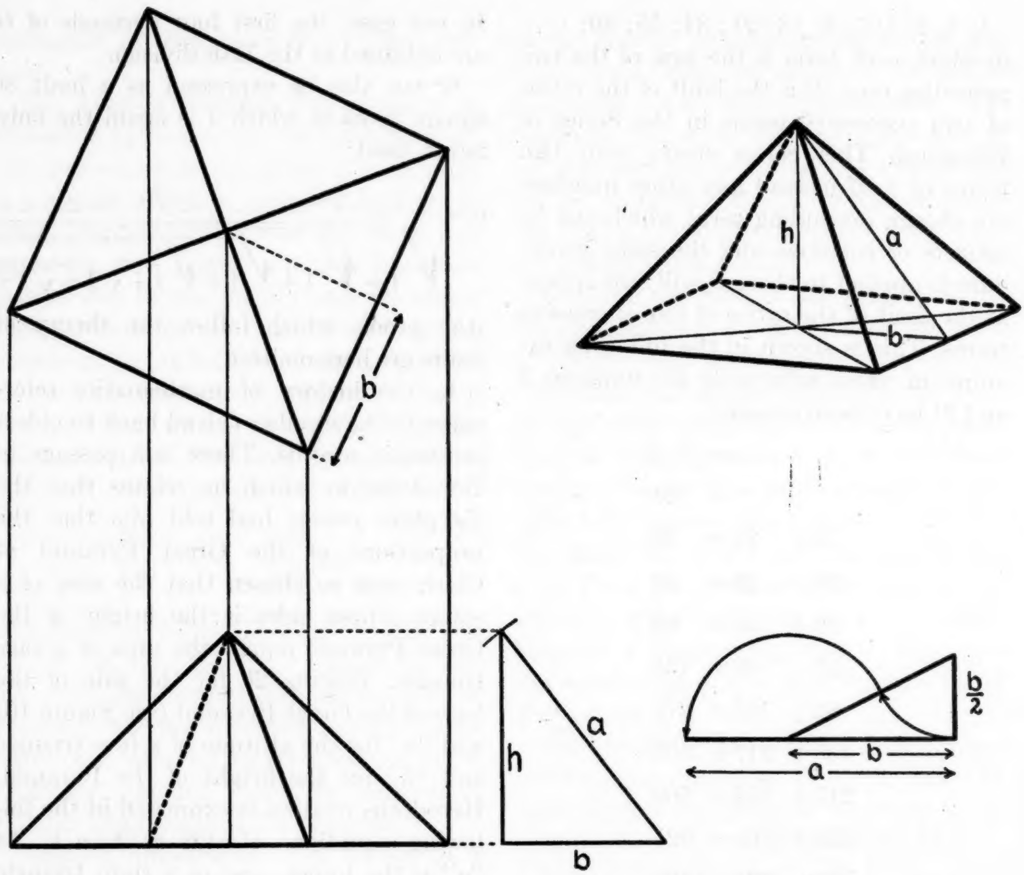


FIG. 10. Construction of the form of the Great Pyramid.

Comparing with $G=0.6180\cdots$ the difference is $0.0002\cdots$.

A further consequence of the statement of Herodotus is the fact that $G$ also appears as the ratio of the base to the lateral area of the Great Pyramid. The sum of the areas of the four face triangles of the great Pyramid is $4\cdot(2b\cdot a)/2=4ab$. The area of the base is $(2b)^2=4b^2$. The ratio of the areas is therefore,

$$\frac{4b^2}{4ab}=\frac{b}{a}=G.$$

The ratio $G$ can be used to construct the form of the Great Pyramid. In Figure 10 first the ground plan of the Pyramid has been drawn. It is a square with its diagonals. Then the elevation is drawn with the positions of the base vertices determined through vertical lines dropped down from the corresponding points of the ground plan. What remains to be drawn is the height of the Pyramid. The altitude of a lateral face is $(1/G)\cdot b$ repreating the construction for $G$ as described before using $b$ as the base, one obtains $b\cdot G$. Adding $b\cdot G$ to $b$ furnishes $b(G+1)=b1/G=a$. Using "$a$" as the hypotenuse and $b$ as one leg of a right triangle, the length of the second leg is the height of the Pyramid $h$. Thus the elevation can be completed. The third projection (upper right in Figure 10) has been obtained from the ground plan and elevation through the methods of descriptive geometry, (described in THE MATHEMATICS TEACHER, April 1946).

In the sixteenth century (1509) Paciolo di Borgo wrote his treatise "DeDivina Proportione" (Of the Divine Proportion) on the ratio $G$. Kepler refers to it as "sectio

divina" (divine section) and Leonardo da Vinci as "sectio aurea" (the golden section) which is a term still in use for it. In an extensive literature on The Golden Section, numerous facts have been collected which show its appearance in forms of nature and art. Hambidge based on it his aesthetic research on "Dynamic Symmetry." Kepler whose sense of proportional relations led him to his three astronomical laws which are the starting point of modern astronomy speaks of the properties of $G$ in his "Mysterium Cosmographicum de Admirabile Proportione Orbium Celestium" as of those of one of the two "great treasures" of geometry, the second being the Theorem of Pythagoras.

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# Division by a Fraction Made Meaningful

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DIVISION by a fraction or by a mixed number is an operation in arithmetic which many teachers find difficult to make meaningful. Then too, its social usefulness has been questioned by curriculum experts and administrators until teachers now seem less concerned with the process. While not so commonly used as many other operations, division by a fraction is important and essential and does provide an unusually effective thinking situation.

It shall be the purpose of this paper to suggest ways in which the process may be made meaningful, sensible and simple to understand. The following problems illustrate the usefulness of the process and provide a springboard for plunging into the problem. The treatment is especially designed to be helpful for teachers of junior high school mathematics or teachers of business mathematics where students are mature enough to be able to enjoy a bit of clear, rigorous thinking. Some of these ideas can be used for first presentation or review in the elementary grades also.

I. *One size can of pears weighs 1 lb. 8 oz. and sells for 49¢. Another weighs 14 oz. and sells for 29¢. Which is the better buy, assuming equal quality and no waste?*

To solve this problem, one finds the cost per pound or the cost per ounce to get comparative prices.

49¢ ÷ 1 1/2 = 49¢ × 2/3 = 98¢/3 = 32.7¢ per lb.
29c ÷ 7/8 = 29¢ × 8/7 = 232¢/7 = 33.4¢ per lb.

or:

49¢ ÷ 24 = 49¢/24 = 2.04¢ per oz.
29¢ ÷ 14 = 29¢/14 = 2.07¢ per oz.

II. *Shelves needed for a cupboard are 3 ft 9 in. long. How many can be cut from a 12 ft. board? How much is wasted? From a 14 ft. board?*

12 ft. ÷ 3 3/4 ft. = 12 × 4/15 = 48/15
= 16/5 = 3.2 ∴ 3.2 shelves
14 ft. ÷ 3 3/4 ft. = 14 × 4/15 = 56/15 = 3 11/15
= 3.7 + ∴ 3.7 shelves

The 12 ft. board is clearly least wasteful, 0.2 shelf is 0.2 × 15/4 ft. or 9 in. waste. In the other case, 11/15 of a shelf will be 11/15 × 15/4 ft. = 11/4 ft. or 2 3/4 ft. or 2 ft. 9 in .waste.

Similarly:

12 ft. ÷ 3 3/4 ft. = 144 in. ÷ 45 in. = 144/45
= 16/5 = 3.2 ∴ 3.2 shelves
14 ft. ÷ 3 3/4 ft. = 168 in. ÷ 45 in. = 168/45
= 3 33/45 = 3 11/15
or 3.7 shelves

Note naturally the same conclusions. Note also the small fractions used. Suppose the corn weighed 15 oz. or the shelf were 3 ft. 8½ in. The solution would be beyond children who can operate with only halves, thirds, and fourths, except by using the smaller denomination and decimals. Perhaps the latter is the better way to help children think this through. However, the problems illustrate the two meanings of division, partition and measurement or successive subtraction, and the need for the understanding of those meanings as well as for understanding the meaning of division of fractions which will now be emphasized on the basis of this practical setting.

1. Multiplying Dividend and Divisor by the Same Number. If both numerator and denominator of a fraction or both dividend and divisor in any division situation are multiplied by the same number, the resulting fraction or quotient is unchanged in value. 12 divided by 4 equals 3. If both 12 and 4 are multiplied by a number such as 5, the quotient is unchanged. Thus,

12 ÷ 4 = 60 ÷ 20 = 3

Similarly, 12 ÷ 1 1/2 can be changed to an equivalent division by multiplying by 2. Thus, 12 ÷ 1 1/2 = (12 × 2) ÷ (1 1/2 × 2) = 24 ÷ 3 = 8

Better yet, the 1 1/2 or 3/2 could be multiplied by 2/3 to make the divisor more simple. Thus, 12 ÷ 3/2 = (12 × 2/3) ÷ (3/2 × 2/3) = (12 × 2/3) ÷ 1 = 12 × 2/3 = 8. Hence, 12 ÷ 3/2 = 12 × 2/3 = 8.

Suppose this principle be applied to problem II above.

(a) 12 ÷ 3 3/4 = (12 × 4) ÷ (3 3/4 × 4)
= 48 ÷ 15 = 48/15 = 16/5 = 3.2

Or:

(b) $12 \div \frac{15}{4} = \left(12 \times \frac{4}{15}\right) \div \left(\frac{15}{4} \times \frac{4}{15}\right)$

$$= 12 \times \frac{4}{15} \div 1 = 12 \times \frac{4}{15}$$

In (a) division is simplified by multiplying both dividend and divisor by the denominator of the divisor or, possibly, by the common denominator if two fractions had been involved. In (b) the factor used is the reciprocal of the divisor. This simple reasoning shows why the correct answer is obtained through multiplying the dividend by the reciprocal of the divisor, and also a closely related method which is widely used in practice. Both depend upon the same general principle; both dividend and divisor can be multiplied by the same number and the quotient will be unchanged.

2. Using a Common Denominator. One meaning of division is "measurement" or successive subtraction. For example, "6 in. ÷ 2 in. = 3," means, "how many lengths 2 inches long can be measured off or subtracted from a length 6 inches." The denominations must be the same before division can be done.

$$6 \text{ ft.} \div 2 \text{ in.} = 72 \text{ in.} \div 2 \text{ in.} = 72/2 = 36.$$

When the denominations are the same, they may be ignored and the division made. Similarly,

$$12 \div 1\,1/2 = 24/2 \div 3/2 = 24/3 = 8$$

"24 halves divided by 3 halves" is treated like 24 inches divided by 3 inches.

As a short cut, how can $12 \div 3/2$ become 24/3 or 8? Apparently,

$$12 \div \frac{3}{2} = \frac{12 \times 2}{3} \quad \text{or} \quad 12 \times \frac{2}{3}.$$

To make the 12 into halves, it is multiplied by the denominator of the divisor. Finally, to effect the division, the denominators are dropped and 24 divided by the numerator 3. This changing the dividend and divisor to fractions with equal denominators reveals the technique of dividing fractions and can be shortened by omitting all except the computational steps. Note the following example:

$$6\,1/2 \div 1\,2/3 = 13/2 \div 5/3$$
$$= 39/6 \div 10/6 = 39/10.$$

Therefore, $13/2 \div 5/3 = 39/10$. The 39 came from $13 \times 3$, and the 10 from $2 \times 5$, therefore, as a short cut,

$$\frac{13}{2} \div \frac{5}{3} = \frac{13}{2} \times \frac{3}{5} = \frac{39}{10}.$$

This reveals the rule which gives the minimum of computational steps.

3. Series. The following series reveals the nature of division and shows why and how the quotient changes as the divisor changes.

$32 \div 8 = 4$

$32 \div 4 = 8$

$32 \div 2 = 16$

$32 \div 1 = 32$

$32 \div 1/2 = ?$

$32 \div 1/4 = ?$

$32 \div 1/10 = ?$

As the divisor decreases, the quotient must increase because there are more of the smaller parts in 32. There are more inches in a mile than there are rods. There are more 2's in 32 than there are 8's. There are more 1/2 minutes in 32 minutes than there are whole minutes, in fact just twice as many. Therefore, $32 \div 1/2 = 64$, $32 \div 1/4 = 128$, and $32 \div 1/10 = 320$.

Similarly, as the divisor increases, the quotient must decrease in the same ratio, but inversely.

$32 \div \frac{1}{10} = 320$

$32 \div \frac{2}{10} = \frac{320}{2}$ since the divisor is twice as large.

$32 \div \frac{3}{10} = \frac{320}{3}$ since the divisor is three times as large.

$32 \div \frac{4}{10} = \frac{320}{4}$ etc.

Again a shortcut will reveal that since $32 \div 4/10$ now equals $320/4$, then

$$32 \div \frac{4}{10} = 32 \times \frac{10}{4} = \frac{320}{4}.$$

This series method and the study of related divisions in series shows satisfyingly the background for the division process and the source of the numbers which make the quotient.

4. Division by Reverse Thinking or Proof. $12 \div 3/2$ must give an answer such that if the answer were multiplied by $3/2$, the product would be 12. Any dividend equals the product of its divisor and quotient. Suppose we use a letter $A$ to represent the answer. Since $12 \div 3/2 = A$, then $A \times 3/2 = 12$. If $A = 12 \times \frac{2}{3}$, then $(12 \times \frac{2}{3}) \times \frac{3}{2}$ will equal 12. Consequently, $A$ may equal $12 \times \frac{2}{3}$. Therefore, substituting for $A$

$$12 \div \frac{3}{2} = 12 \times \frac{2}{3} = \frac{24}{3} = 8.$$

Prove it correct: $8 \times 3/2 = 12$; quotient times divisor equals dividend.

Suppose we use a second illustration: $6\ 1/2 \div 1\ 2/3 = ?$ Or, $13/2 \div 5/3 = A$, some answer such that $A \times 5/3 = 13/2$. Then $A$ may be $13/2 \times 3/5$, because $A \times \frac{5}{3} = \frac{13}{2}$ and $(\frac{13}{2} \times \frac{3}{5}) \times \frac{5}{3} = \frac{13}{2}$. Therefore, substituting for A

$$\frac{13}{2} \div \frac{5}{3} = \frac{13}{2} \times \frac{3}{5} = \frac{39}{10}.$$

Proof: $39/10 \times 5/3 = 195/30 = 13/2 = 6\ 1/2$.

5. Graphic Representation of Division by a Fraction. Any problem such as $12 \div 1\ 1/2$ can be represented graphically by line segments, circles, or areas. How many periods 1 1/2 hr. in length can be made from 12 hr.? How many articles costing \$1 1/2 each can be purchased for \$12? How many strips each 1 1/2 ft. long can be cut from a strip 12 ft. long? Twelve circles, or a line segment 12 units long or a rectangle 12 units long can be used to show the nature of the process in problems of this kind. For example, a line $AB$ is 12 units long. If it is cut into pieces 1 1/2 units long, it is evident that in every 3 units there are 2 pieces; $12 \div 3 = 4$ and $4 \times 2$ pieces $= 8$ pieces. Or, $12 \div 1\ 1/2 = 12 \div 3 \times 2 = 12 \times 2/3 = 8$. This can readily be illustrated graphically, and, to save expense, need not be so done here. Similar concrete or graphic evidence is revealed in 12 circles, each 3 of which would provide 2 groups composed of 1 1/2 hours, dollars, miles, or any other unit. And consequently, $12 \div 1\ \ 1/2 = 12 \div \frac{3}{2} = \frac{12}{3} \times 2$ or $12 \times \frac{2}{3}$.

To generalize this thinking, let us use another problem, $6\ 1/2 \div 1\ 2/3$. Graphs suggest that the denominator 3 reveals that 3 pieces require $3 \times 1\ 2/3$ or 5 units which may be feet, hours, dollars, or any other denomination. Therefore, each 5 units now provides 3 pieces or groups and consequently,

$$6\ 1/2 \div 1\ 2/3 \text{ or } 6\ 1/2 \div 5/3 = \frac{6\ 1/2}{5} \times 3$$

$$\text{or } 6\ 1/2 \times \frac{3}{5} \text{ or, directly,}$$

$$\frac{13}{2} \div \frac{5}{3} = \frac{13}{2} \times \frac{3}{5}.$$

It is the opinion of the author of this article that the graphic analysis is used far too little. Possibly some slides or film strips could be made which would assist materially here. The possibilities for clarity and concreteness are worthy of extensive exploration.

6. Laboratory Experimentation, Using Actual Measures. Suppose one were to start with a board 12 ft. long or some ribbon 12 yd. long or \$12 or 12 qt. or 12 hr. and one wished to obtain pieces 1 1/2 ft or 1 1/2 yd. long, etc. If they are actually measured off, it would be evident again as with the line segment, that for each 3 ft. or 3 yd. or 3 hr. there would be 2 pieces or, 2/3 as many pieces as feet or yards or other units. Therefore, $12 \div \frac{3}{2} = 12 \times \frac{2}{3}$. Or, for each 3 ft. or hr. there would be 2 pieces. Consequently, $12 \div \frac{3}{2} = \frac{12}{3} \times 2 = 12 \times \frac{2}{3} = 8$. Finally, the eight pieces could actually be cut and laid out to show that

they are the same length and that each one is 1 1/2 ft. long.

A similar demonstration could be made or imagined with 12 apples, \$12, 12 hours, 12 qt. or, naturally, with any other dividend and divisor; for example, in 14 ft ÷ 3 3/4 ft. Again here, the denominator reveals the number of pieces in a certain length; that is, 4 pieces here would require 15 ft. Then there would be 4 pieces for each 15 ft. or 4/15 as many pieces as feet. Consequently, $14 \div 3\ 3/4 = \frac{14}{15} \times 4$ or $14 \times \frac{4}{15} = 3\ \frac{11}{15}$. Therefore, $14 \div \frac{15}{4} = 14 \times \frac{4}{15}$.

7. Analysis. Probably the most common approach used in textbooks is analysis. To me, it seems the most difficult and mature, and in many ways the least fruitful. It might start with whole numbers, then go to a unit fraction divisor, then to any fraction as a divisor and finally to any dividend. 12 ÷ 3 may mean "how many times can 3 be subtracted from 12." The units involved may naturally be any units. The meaning may also be, "how many in each part if 12 is divided into 3 parts." The first meaning is preferable because division abandons its meaning as partition when the divisor is a fraction. 12 in. ÷ 3 in. = 4, and 12 in. ÷ 3 = 4 in., and also 12 in. ÷ 1/3 in. = 36, but the statement, "12 in. ÷ 1/3 = 36 in." has little meaning, if any, since 12 in. cannot be divided into 1/3 of a part with 36 in. in each part except by mental gymnastics. To show that $12 \div 3 = 12 \times 1/3$ or 4, is not difficult. Then let's tackle $12 \div 1/3$. But first $1 \div 1/3 = 3$ since there are 3 thirds in 1 unit. Consequently, since $12 = 12 \times 1$, then $12 \div 1/3 = 12 \times (1 \div 1/3) = 12 \times 3 = 36$. Similarly, any number divided by 1/3 equals that number $\times 3$, since $1 \div 1/3 = 3$. Hence, $6\ 1/2 \div 1/3 = 6\ 1/2 \times 3 = 39/2$.

Now, the remaining problem is to analyze the effect of changing the numerator of the divisor; division by a unit fraction is easy. First one needs to establish the general principle that if one doubles the divisor the quotient will be half as large. Or, if one multiplies the divisor by any number, $n$, one will divide the quotient by the same number, $n$.

$$(a)\ 48 \div 2 = 24,\ (b)\ 48 \div 4 = 12,$$

$$\text{or } (c)\ 48 \div 6 = 8$$

in ($b$) the divisor is twice and in ($c$) three times that of ($a$). The quotient $n(b)$ is 1/2 of what it is in ($a$), and in ($c$) 1/3 of what it is in ($a$). There if

$$12 \div 1/3 = 36, \text{ then } 12 \div 2/3 = 36/2 = 18$$

$$\text{or } 12 \div 2/3 = \frac{12 \times 3}{2} \text{ or } 12 \times 3/2.$$

Further illustration:

$$12 \div 3\ 3/4 \text{ or } 12 \div 15/4 = ?$$

First,

$$12 \div 1/4 = 12 \times 4 = 48, \text{ since } 1 \div 1/4 = 4.$$

Then,

$$12 \div 15/4 = \frac{12 \times 4}{15} \text{ or}$$

$$12 \times 4/15 = 48/15 = 3.2.$$

These settings for giving substance, meaning, or common sense to the process of dividing by a fraction or a mixed number are by no means the only ones. They are suggestive, however, and have been used many times with satisfying results. It would be interesting to have teachers try them out and report on results or to report on other settings or analyses which have been found useful.

# THE ART OF TEACHING

## An Idea

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### I. WHY AN IDEA?

IN RECENT years it has become fashionable to recommend to teachers in the elementary schools and the high schools the organization of their mathematics offerings "into a few large units built around key concepts and fundamental principles."[1] In a previous paper[2] the writer claimed that mathematics teachers were in no position to follow such recommendations.

At the very least, a useable set of "key concepts and fundamental principles" must be formulated by an authoritative group. Teachers must be assured that by teaching these "ideas" they will be giving their students the best preparation for college and for life which it is practicably feasible to provide under the present organization of our schools. Until this is done the teachers in the public schools will continue to teach topics. Many teachers will feel impelled to cover all the sections of a conventional text in a sincere effort to provide the students with the best preparation for college. Other teachers, especially in the lower grades, will endeavor to teach "functional mathematics," that is to cover topics like taxation and insurance in a way to make the knowledge and skill acquired directly useful to the boys and girls as they grow into adult members of the social group. This writer agrees with the thinking of contemporary committees in concluding that teaching by topics as interpreted in either of these two ways fails miserably to solve the problems of the mathematics curriculum in the public schools. Hence, he feels keenly the need for the formulation of a set of key concepts and fundamental principles.

To date there has been no such formulation which mathematics teachers over the nation could follow. On the level of general principles there have been some excellent statements made in committee reports, methods tetxts, and articles appearing in various periodicals. There is a pressing need to implement these general statements with enough particulars to enable teachers to put them to work.

As a member of a committee charged with writing a teachers manual for a new state course of study in Iowa, the writer faced this problem. The "idea" which he had chief responsibility for developing was "comparing quantities." The idea is not new and certainly not original with the writer. Even the development of the idea was a committee[3] responsibility rather than an individual one. This development is presented in an abbreviated form below because the writer feels that it illustrates a pattern which might be followed in presenting a list of key concepts and fundamental principles. Moreover, in a way typical of committees, several members have already moved. The thinking of the committee cannot possibly be preserved in full. The writer feels so strongly about

[1] The Second Report of the Commission on Post War Plans, MATHEMATICS TEACHER, May 1945, p. 209.

[2] Interpretation of College Preparation by Individual Teachers of High School Mathematics, MATHEMATICS TEACHER, December 1947, p. 377.

[3] Committee on mathematics for grades 7, 8, and 9. Miss Dorothy Horn, Woodrow Wilson Junior High School, Des Moines, Iowa; Miss Chrystal Kolpin, Webster City High School, Webster City, Iowa; Mr. Paul Seydel, Principal, Sheldon High School, Sheldon, Iowa; Dr. H. C. Trimble, Iowa State Teachers College, Cedar Falls, Iowa. (1946–47 addresses given.)

the values of this thinking that he wishes to preserve and give publicity to some small part of it.

The four theses of this paper are:

1. The mathematics curriculum should be built around a few key concepts and fundamental principles, that is, "ideas."
2. One such idea is "comparing quantities."
3. The pattern followed in developing this idea below gives a minimum of particulars needed to make a general idea useable by mathematics teachers over the nation.
4. There is a pressing need for such definite statements by an authoritative group.

## II. The Idea

The need to compare quantities is common to all boys and girls in grades seven through nine whether or not these boys and girls intend to pursue the study of mathematics farther. The idea of comparison can be used effectively as one key concept around which to build the mathematics curriculum of grades seven through nine. In relation to this central idea one can teach more about fundamental operations, introduce the language of percent, and make use of the various ways of stating relationships as an aid to solving the problems which arise naturally in connection with comparing quantities.

### II.-1 *Content and procedures for the seventh grade unit on comparing quantities*

Boys and girls in grade seven should be helped to discover the need for the two ways of comparing quantities. Vague ideas about comparison by subtraction and comparison by division should be sharpened and focused.

Since comparison by subtraction is relatively easy, one should begin at this point. Applications of increasing complexity should be given to establish the concept as one of broad, general usefulness. The language of such comparisons should be mastered for its inherent value and for its value as a preparation for the language of comparison by division.

The introduction to comparison by division should be through problems involving readily perceived number relationships. "John is twice as old as Mary." Equivalent statements, "Mary is half as old as John," should become equivalent in the minds of the students.

As these comparisons become more involved two aids to thinking will be needed

a. Translation of verbal statements into simple equations
b. Ratios leading to proportions.

In light of *a* above, the problem—"One fourth of the crowd at the game paid adult admissions. If fifty-four adult admissions were paid, how large a crowd saw the game?"—becomes—"One fourth of what number is 54?"—and finally—"$\frac{1}{4}\cdot n=54$." From this point of view the "three cases" of per cent become all one. In light of *b* above the same problem becomes—"one adult for every four in the crowd"—"$\frac{1}{4}=\frac{54}{n}$".

The time needed to teach "translation" and "proportion" will depend upon the background of the students. If workbooks used in earlier grades used $n+4=7$, then $n=$____. rather than____$+4=7$, the job of teaching translation will be easier. If $\frac{1}{4}$ means only, cut a whole in 4 equal parts and take one, it will be necessary to establish the idea, for every four take one.

It is essential to lay the groundwork well in grade seven. Time should be taken to overcome conceptual difficulties. Too many high school graduates do not have the ratio concept. Without this concept much of the idea (comparing quantities) is lost. The statement "three fourths of the students were girls" means

| Girls | Boys | Class |
|---|---|---|
| . . . | . | . . . . |

etc.

Many ways of tying the ratio idea to the nerves and muscles of students should be

used until the idea becomes part of their thinking equipment.

When translation and ratio have been mastered, and when the language of comparisons comes easily, the additional language of per cent will no longer present an insurmountable barrier to so many students. "Three fourths of the students were girls"—75% girls, 25% boys—these are easy deductions in light of an understanding of ratio.

Problems of per cent increase and decrease are to be studied in greater detail in grade eight. A beginning should be made in grade 7. 20% more means for every 100 to begin with there will now be 120. Problems like "8% less than a number is 63, find the number become either

$$n-.08n=63 \quad \text{or else} \quad \frac{92}{100}=\frac{63}{n}\cdot$$

It is usually easier to teach the translation approach to these harder problems. The work in grade 7 should be of a preparatory nature and should stop short of the point of confusion. It is much more important to thoroughly establish the basic methods at this level than to attack the more difficult problems.

II.-2 *Content and procedures for the eighth grade unit on comparing quantities*

The concepts encountered in the seventh grade should be strengthened and extended in grade 8.

The combination of comparison by subtraction and comparison by division has frequently proved to be too much for the boys and girls. The language subtleties of problems of per cent increase and decrease have presented a formidable barrier. This barrier can be overcome if it is recognized. The methods of translation and of ratio will help.

Problems involving increase and decrease by a definite amount should be studied first. In the problem—"Jack made 21 points in the game. If he made five points in the first quarter and increased his total by nine points in the second quarter, how many points did he make in the second half"—should be met. The importance of changing "by" to "to" and hence making the answer 12 rather than 7 should be noted. The boys and girls should be made conscious of the precision of language required for the correct interpretation of a problem before they begin the study of per cent increase and decrease.

With this preparation, a problem like—"The cost of lighting the school auditorium was cut 30% by installing fluorescent fixtures. If the bills averaged $24 during the previous year, what should they average during the following year"—should present no serious barriers. The two forms of solution, namely

$$24-(.30)(24)=n$$

and

$$\frac{\text{old cost}}{\text{new cost}}=\frac{100}{70}=\frac{24}{n}$$

should be used. Generally speaking, the first form is suited to more difficult problems requiring precise answers while the second form is suited to easier problems and to estimating answers mentally.

Other problems may, because of the difficulty of visualizing the relationships involved, still prove too difficult for some or all students. The opportunity to acquire some more of the aids to relational thinking should be welcomed at this point.

When the teacher uses a dot diagram to help the student visualize the fraction 3/4 as a ratio, this is an instance of using graphs or charts to clarify a relationship. When 3/4 is seen as

| *a* | *b* | *c* |
|---|---|---|
| · · · | · | · · · · |
| · · · | · | · · · · |

it becomes clear the $\frac{3}{4}=\frac{6}{8}=\cdots$ and that finding $\frac{3}{4}$ of something means dividing it in the ratio 3 to 1.

When the teacher recognizes and uses the four ways of stating a quantitative relationship as they are needed to help boys and girls to better visualize such a relationship she is beginning to teach relational thinking.

When one reads: "The power of an engine is increased 20% by installing a super-

charger" this statement can be paraphrased in many ways.

The teacher should have available

(1) Verbal paraphrases
(2) Tables
(3) Graphs or charts
(4) Equations or formulas.

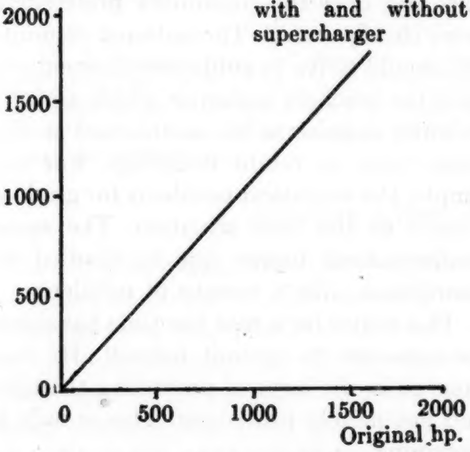
If the question—"What initial power is needed to produce, with the addition of a supercharger, a 2000 hp engine?"—causes trouble, a further study of the relationship of 20% increase may be needed. As examples of ways to get the needed insights, consider

(1) 20% increase means to each 100 hp of the original engine the supercharger adds 20 hp and then the engine has 120 hp.

(2)

| Original hp | Added hp | New hp |
|---|---|---|
| 100 | 20 | 120 |
| 200 | 40 | 240 |
| | etc. | |

(3) New hp. Original and final hp. with and without supercharger




(4) New hp = original hp + 20% of original hp.
i.e. $H = h + .20h$
or $H = 1.20h$

It is suggested that the teacher make use of every available aid to relational thinking. The required skills of constructing tables, line graphs, and simple formulas should be taught as needed. This makes these skills secondary to the main problem of comparing quantities (the new to the old horsepower in the problem above). This does not mean that the teacher should assume that the boys and girls have these skills. Rather the skills should be taught as they are needed and to the degree of perfection needed for making them helpful in connection with comparing quantities.

The very important question of comparing groups of quantities has not been mentioned. A unit on comparison of groups will be placed in ninth grade general mathematics. To the extent that time is available and the students are ready for it an introduction to this question may be given in grade 8.

### II.-3 *Content and procedures for the ninth grade unit on comparing quantities*

With the idea in mind that a general mathematics program for grade 9 should be a continuation of the offerings of the previous grades, this unit is designed to extend the ideas presented in grades 7 and 8. The following outline is shortened to keep this discussion to a reasonable length.

Under the title "comparing groups," the unit explores the difficulties of expressing a characteristic of a group as a single number. Averages are used with all due precaution to compare two or more groups.

Rather than make this unit a formal, though elementary, course in statistics, it is centered around a few selected problems which are inherently interesting to the boys and girls, provide opportunities for the teacher to raise significant statistical questions, and yet simple enough to permit reaching some definite conclusions based upon the analysis. It is considered more important to lead the boys and girls to an appreciation of the importance of statistical comparisons, and to constructively critical attitudes toward statistical reasoning than to teach them some of the skills needed by a statistical clerk. Such skills can be acquired on the job by a person who has good fundamental training in arithmetic and an appreciation of the fact that numbers can be either instructive or misleading depending upon the manner in which they are analysed and interpreted.

A problem which has often proved fruitful for these purposes is the "Am I normal"

question. This is an area of tension for many boys and girls and a subject of considerable interest for all. Data on heights, weights, head measurements, and the like can be obtained from measurements made in the classroom or from secondary sources. Questions related to the size of the sample, choice of an appropriate average, and definition of "unusual" deviations from the average chosen, are met. Frequency tables are made and translated into column diagrams and broken line graphs. Methods of calculating the arithmetic mean provide opportunities for maintaining and extending computational skills. The median and mode are introduced and the special advantages of each "average" noted. The quartile range is used as a measure of the spread of the distribution.

Other problems are introduced as needed for extending the desired concepts and providing some drill. Groups are compared as to average and spread about the average. There is a good opportunity to demonstrate the need for quantitative thinking as a basis for intelligent citizenship as well as for participating in a vocation.

### III. Putting the Idea to Work

Although the development of the idea of comparing quantities has been presented in some detail, much remains to be done before a busy public school teacher can and (or) will use it.

Passing from topic to unit organization is a radical change which needs the support of an authoritative group. Individual teachers need such backing before they will feel confident in adopting even a plan which appeals to them.

A single idea might be stretched to cover much of the subject matter of mathematics for grades 7, 8, and 9. Some writers have used the idea of functional thinking in this way. This writer prefers to have several key ideas stated by a national committee and to have these key ideas defined by enough particulars to make them meaningful. The list of key ideas recommended should be comprehensive in the sense that an acceptable curriculum could be based upon them. The Iowa committee thought of the units on comparing quantities as comprising one fourth to one third of the work of grades seven through nine.

Finally, more details should be provided. It seems reasonable to assume that textbook writers would eagerly undertake this task once the consumers' preferences were clearly stated. The national committee should strive to guide developments to give the teachers materials which are sufficiently definite to be useable and at the same time to retain flexibility. For example, the statistical problems for grade 9 should fit the local situation. The same mathematical topics can be studied in connection with a variety of problems.

This writer feels that the time has come for someone to commit himself. He has done so in the hope of provoking thought and not merely provoking. This is only a beginning.

# EDITORIALS

## Reorganization of Secondary Education

SEVERAL prominent educators* in recent articles or at educational meetings have emphasized the need for a revision of the secondary school curriculum to bring it closer to the lives of the approximately 7,000,000 youth now preparing for college or for skilled professions. It has been said that eight out of ten pupils now entering the fifth grade also enter the ninth, but only four of the eight finish high school. The problem thus becomes one of planning the curriculum in such a way as to hold youth in the high school and even in the junior colleges longer, and in greater numbers than is true at the present time.

* U. S. Commissioner of Education John Studebaker, Homer P. Rainey, and others. See also "Citizens Look at Education," A Progress Report by The Citizens Federal Committee on Education, 1947–48.

The problem for the National Council of Teachers of Mathematics to consider is, to what extent does mathematics need reorganization to keep abreast of the changes and adjustments that are made in general. What is there in the present offering in mathematics that needs to be revalued and how can the content be so reorganized and taught as to be worth while in the lives of the boys and girls who make up the secondary school? It is time for action.

What has been said above is in no sense a criticism of the forward looking things that have already been done by the Council, by the Joint Commission, the Commission on Post-War Plans in Mathematics, and the like. The point is that some kind of reorganization of the curriculum should be going on all the time to meet the demands of the changing times.—W. D. R.

## Guidance Pamphlet in Mathematics

THE MATHEMATICS TEACHER is reprinting a large number of the final report of the Commission on Post-War Plans in Mathematics that appeared in the November (1947) issue of the magazine, is giving the report a substantial cover, and will call it "A Guidance Pamphlet in Mathematics," to be used by teachers, guidance counselors, and pupils to help the pupils to decide how much mathematics they should elect in their school programs in order not to be handicapped in their chosen field. It will undoubtedly be a fine thing if organizations of mathematics teachers all over the country order these reprints in such numbers as they can afford, and distribute them widely where they will do the most good. In order to assist as much as possible, THE MATHEMATICS TEACHER will sell these reprints in quantities of ten or more to such groups, at 10¢ each postpaid. However, it will be of great assistance to the office of THE TEACHER if these orders are sent in promptly and accompanied by the correct amounts involved. A large number of orders has already come in, and it is quite necessary to know in advance how many more will be needed to supply the demand.—W. D. R.

# NEW BOOKS

1. Bell, E. T., *The Development of Mathematics.* McGraw-Hill Book Co., Inc., New York, 1945. 600 pp. $5.00.
2. Bell, E. T., *The Magic of Numbers.* Whittlesey House, McGraw-Hill Book Co., Inc., New York, 1946. 418 pp. $3.50.
3. Bell, Clifford, and Thomas, Tracy Y., *Essentials of Plane and Spherical Trigonometry* (Revised Edition). Henry Holt & Co., New York, 1946. 350 pp. $2.30 with tables, $2.00 without tables.
4. Braverman, Benjamin, *Gaining Skill in Arithmetic.* D. C. Heath & Co., Boston, 1945 134 pp. $1.40.
5. Campbell, Harold G., and Wren, F. Lynwood, *Number Readiness Series* (Revised Edition). D. C. Heath & Co., New York, 1947. $1.20.
   *Exploring Numbers*, 264 pp.; *Discovering Numbers*, 280 pp.; *Number Activities*, 247 pp *Number Experiences*, 248 pp.
6. Coolidge, J. L., *A History of the Conic Sections and Quadric Surfaces.* Oxford University Press, 1945. 214 pp.
7. Cornett, R. Orin, *Algebra. A Second Course.* McGraw-Hill Book Co., Inc., New York, 1945. 313 pp. $2.00.
8. Curtis, Arthur B., and Cooper, John H., *Mathematics of Accounting.* Prentice-Hall Inc., New York, 1947. 550 pp.
9. DeGroat, Harry De W., and Young, William E., *Iroquois New Standard Arithmetics.* Iroquois Publishing Company Inc., Syracuse, New York, 1945. Grades 3–8; with answer book.
10. Eddington, Sir A. S., *Fundamental Theory.* Cambridge University Press, The Macmillan Company, New York, 1946. 292 pp. $6.00.
11. Edwards, William Herbert, *Precision Shop Mathematics.* D. C. Heath & Co., Boston, 1947. 314 pp. $2.48.
12. Ewing, Claude H., and Hart, Walter W., *Essential Vocational Mathematics.* D. C. Heath and Co., Boston, 1945. 266 pp. $1.60.
13. Granville, William Anthony, Smith, Percey F., and Longley, William R., *Elements of Calculus.* Ginn & Co., Boston, 1946. 549 pp. $3.75.
14. Graves, Lawrence M., *The Theory of Functions of Real Variables.* First Edition. McGraw-Hill Book Co., Inc., New York, 1946. 300 pp. $4.00.
15. Hart, Walter W., and Jahn, Lora D., *Mathematics In Action.* D. C. Heath & Co., New York, 1947. Book one, 340 pp. Book two, 324 pp. Book three, 438 pp.
16. Hart, William L., *College Algebra*, Third Edition. D. C. Heath & Co., Boston, 1947. 415 pp.
17. Jeffreys, Harold, and Jeffreys, Bertha Swirlés, *Methods of Mathematical Physics.* Cambridge University Press, The Macmillan Co., New York, 1946. 679 pp. $15.00.
18. Keniston, Rachel P., and Tully, Jean, *Plane Geometry.* Ginn & Co., Boston, 1946. 392 pp.
19. Knight, F. B., Studebaker, J. W., and Tate, Gladys, *Mathematics and Life, Book 2.* Scott, Foresman & Co., Chicago, 1946. 512 pp.
20. McKay, Herbert, *The World of Numbers.* The Macmillan Co., New York, 1946. 198 pp., $2.50.
21. Mallory, Virgil S., *New Trigonometry.* Benjamin H. Sanborn & Co., Chicago, 1947. 264 pp. $2.00.
22. Maxwell, E. A. *The Methods of Plane Projective Geometry Based on the Use of General Homogeneous Coordinates.* Cambridge, at the University Press, The Macmillan Co., New York, 1946. 230 pp. $2.75.
23. Miller, Earle B., *Intermediate Algebra for Colleges.* Ronald Press Co., New York, 1947. 361 pp. $2.50.
24. Miller, Frederick H., *Calculus.* Second Edition. John Wiley & Sons, Inc., New York, 1939, 1946. 416 pp.
25. Morrill, William Kelso, *Plane Trigonometry.* Revised Edition. Rinehart & Co., Inc., New York, 1946. 245 pp. $2.50.
26. Murnaghan, Francis D., *Analytic Geometry.* Prentice-Hall, Inc., New York, 1946. 402 pp. $3.25.
27. Murray, Francis J., *The Theory of Mathematical Machines.* King's Crown Press, New York, 1947. 116 pp. $3.00.
28. Nelson, Alfred L., Folley, Karl W., and Borgman, William M., *Calculus.* Revised Edition. D. C. Heath & Co., Boston, 1946. 376 pp.
29. Newsom, Carroll V., *Introduction to College Mathematics.* Prentice-Hall, Inc., New York, 1946. 337 pp.
30. Nicholson, Fred, *Mechanical Drawing.* D. Van Nostrand Co., Inc., New York, 1946. 205 pp. $2.00.
31. Northcott, John A., *Mathematics of Finance.* Rinehart & Co., Inc., New York, 1946. 250 pp.
32. Nowlan, Frederick S., *Analytic Geometry.* Third Edition. McGraw-Hill Book Co., Inc., New York, 1946. 355 pp. $2.75.
33. Nyberg, Joseph A., *Fundamentals of Solid Geometry.* American Book Co., New York, 1947. 267 pp. $1.72.
34. Orleans, Joseph B., and Hart, Walter W., *Intermediate Algebra.* Second Edition. D. C. Heath & Co., Boston, 1947. 296 pp.

35. *Proceedings of the First Canadian Mathematical Congress*, 1946, The University of Toronto Press. $3.25, 364 pp.
36. Randolph, John F., and Kac, Mark, *Analytic Geometry and Calculus*. The Macmillan Co., New York, 1946. 642 pp. $4.75.
37. Rasch, William E., *Practical Electrical Mathematics*. D. C. Heath & Co., Boston, 1946. 357 pp. $2.00.
38. Reddick, H. W., and Miller, F. H. *Advanced Mathematics for Engineers*. Second Edition. John Wiley & Sons, Inc., New York, 1947. 508 pp. $5.00.
39. Rider, Paul R., *Analytic Geometry*. The Macmillan Co., New York, 1947. 383 pp.
40. Rietz, H. L., Crathorne, A. R., and Adams, L. J., *Intermediate Algebra*. Revised Edition. Henry Holt & Co., New York, 1947. 294 pp.
41. Schorling, Raleigh, and Clark, John R., *Mathematics In Life*. World Book Co., 1946. 500 pp. $1.80.
42. Slade, Samuel, and Margolis, Louis, *Mathematics for Technical and Vocational Schools*. Third Edition. John Wiley & Sons, Inc., New York, 1922, 1936, 1946. 532 pp. $2.50.
43. Smith, Edward S., Salkover, Meyer, and Justice, Howard K., *Unified Calculus*. John Wiley & Sons, Inc., New York, 1947. $3.50. 507 pp.
44. Tuites, Clarence E., *Basic Mathematics for Technical Courses*. Prentice-Hall, Inc., New York, 1946. $5.00. 132 pp.
45. Upton, Clifford B., and Fuller, Kenneth G., *Arithmetic, Grade Six*, and *Arithmetic, Grade Seven*. American Book Co., New York, 1947. Gr. 6, 312 pp., $1.20. Gr. 7, 328 pp., $1.24.
46. Van Ruyl, George H., *Business Arithmetic*. American Book Co., New York, 1947. 352 pp.
47. Watts, Earle F., and Rule, John T., *Descriptive Geometry*. Prentice-Hall, Inc., New York, 1946. 301 pp. $3.00.
48. Weil, André, *Foundations of Algebraic Geometry*. American Mathematical Society, New York, 1946. 288 pp.
49. Wertheimer, Max, *Productive Thinking*. Harper & Bros., Publishers, New York, 1945. 224 pp. $3.00.
50. Wolfe, John H., Mueller, William F., and Mullikin, Seibert D., *Industrial Algebra and Trigonometry, with Geometrical Applications*. First Edition. McGraw-Hill Book Co., Inc., New York, 1945. 389 pp. $2.20.
51. Wren, F. Lynwood, *Number Readiness Series*. Revised Edition. D. C. Heath & Co., 1947. *Number Relations*, 310 pp., $1.24. *Functional Numbers*, 326 pp., $1.28.
52. Wrightstone, J. Wayne, and Meister, Morris, *Looking Ahead in Education*. Ginn & Co., Boston, 1945. 151 pp. $1.50.

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